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VOLUME - II

CHAPTER 1

Gravitation

CHAPTER OUTLINE

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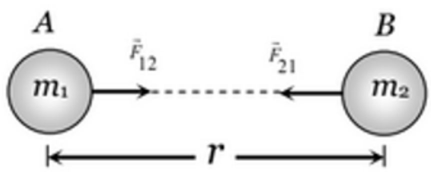
INTRODUCTION

- Every body in the Universe attracts every other body.
- The gravitational force is the attraction between any two bodies in the Universe. It is both a fundamental force and nature's weakest force.
- Any body with mass (whether a very small mass like an atom or a very massive mass like the Sun or stars) exerts gravitational force on any other body with mass.
- As a result, gravitational force is caused by the mass of the interacting bodies. The law that governs the gravitational force between any two bodies in the Universe is known as 'Newton's Universal law of gravity,' after Sir Isaac Newton, who discovered it.

UNIVERSAL LAW OF GRAVITATION

- According to Newton's universal law of gravitation, "every particle in the universe attracts every other particle in the universe with a force that is directly proportional to the product of those particles' masses and inversely proportional to the square of the distance between those particles." This force acts along the line connecting the two particles."



- Consider two particles 1 and 2 having mass 'm_1' and 'm_2' separated by a distance 'r'. According to Newton's universal law of gravity, the gravitational force of attraction between two particles (F) is

  $F \propto m_1 m_2$ and $F \propto \frac{1}{r^2}$

  $\therefore\ F \propto \frac{m_1 m_2}{r^2}$ or

  $$F = \frac{G m_1 m_2}{r^2}$$

  where 'G' is a proportionality constant known as the universal gravitational constant. Particle 1 exerts a force on particle 2 (F_{21}) and it is an attractive force directed towards 1 from 2 Similarly, particle 2 exerts a force on particle and it is an attractive force directed towards 2 from 1 (F_{12}).
- As a result, the gravitational force between 1 and 2 constitutes an action-reaction pair.

  $\vec{F}_{12} = -\vec{F}_{21}$; $F = \left|\vec{F}_{12}\right| = \left|\vec{F}_{21}\right|$, then $F = \frac{G m_1 m_2}{r^2}$

If m1= m2 = 1kg and r=1 m, then

$$F=\frac{Gm_1m_2}{r^2}=\frac{G\times1\times1}{l^2}=G$$

- Thus, the universal gravitational constant 'G' is numerically equivalent to the gravitational force between two particles of unit mass separated by unit distance.
- The value of ' G ' was first experimentally established by Henry Cavendish. The SI unit of ' G is Nm^2kg^{-2} and its dimensional formula is $[M^{-1}L^3T^{-2}]$ The value of $G=6.67*10^{-11}$ Nm^2kg^{-2}
- This law is known as the universal law of gravity because it holds true regardless of the nature of the objects (such as size, form, mass, and so on), as well as in all places and at all times. The value of is unaffected by the mass of the particles, the distance between them, or the medium that separates them.

Characteristics of Gravitational Force

- The gravitational force between any two bodies forms an action-reaction pair. The force exerted on one body by the other body is the same magnitude but in the opposite direction.
- In nature, the gravitational pull between any two bodies is always attractive.
- The gravitational force between two bodies is unaffected by the presence or absence of any other body in their vicinity.
- The gravitational force between two bodies is independent of the medium between them. As a result, protecting a body from gravitational force is impossible.
- Gravitational force is a central force, which means that it works along the line connecting two interacting particles.
- Gravitational force is the weakest force in nature.
- The gravitational force is negligibly modest for light bodies but becomes rather important for massive bodies such as planets, satellites, and stars.
- Gravitational force is a long-range force, which means it is effective even when the distance between the interacting particles is great. The gravitational attraction between the Sun and the planet Pluto, for example, exists despite the great distance between them and is responsible for Pluto's journey around the Sun.
- The force of gravity is a conservative force. As a result, potential energies are linked to gravitational forces.

Notes:

Newton's Universal Law of Gravitation applies only to particles or point masses. When the sizes of the bodies are relatively small in comparison to their separation, they can also be viewed as particles. It may also be demonstrated that a body with spherical symmetry in mass distribution can be considered as a particle, with mass concentrated in the sphere's core exclusively for gravitational interaction at points outside the sphere. If the interacting bodies cannot be reduced to particles, the integration method must be employed to calculate the gravitational force.

Superposition Principle of Gravitational Forces

- The gravitational force between two particles is independent of whether or not other particles are present. This gives rise to the gravitational force superposition principle.
- According to the superposition principle, the gravitational force on a particle of mass $m_1,m_2,m_3.....m_n$, due to a distribution of particles of masses around it, is the vector sum of the gravitational forces exerted on m by each of the other particles , the forces between each pair being independent of the other particles $m_1,m_2,m_3.....m_n$

  For example

  Consider a distribution of six particles (m_1,m_2,m_3,m_4,m_5 and m_6) around a particle of mass as shown

$$\bar{F}_{01} = \frac{-Gmm_1}{|\bar{r}_1|^2}\hat{r}_1 = -\frac{Gmm_1}{|\bar{r}_1|^3}\bar{r}_1$$

$$\bar{F}_{02} = -\frac{Gmm_2}{|\bar{r}_2|^2}\hat{r}_2 = -\frac{Gmm_2}{|\bar{r}_2|^3}\bar{r}_2$$

....

....

$$\bar{F}_{06} = -\frac{Gmm_6}{|\bar{r}_6|^2}\hat{r}_6 = -\frac{-Gmm_6}{|\bar{r}_6|^3}\bar{r}_6$$

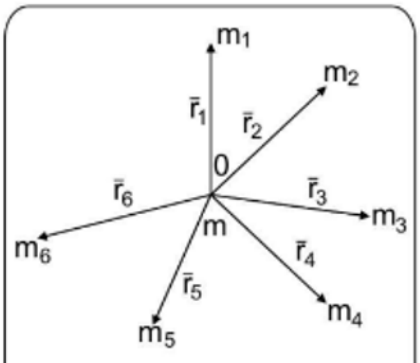


Fig. 3.2

F_G = gravitational force on m due to m_1, m_2, m_3, m_4, m_5 and m_6 as per superposition principle is

$$\bar{F}_a = \bar{F}_{01} + \bar{F}_{02} + \bar{F}_{03} + \bar{F}_{04} + \bar{F}_{05} + \bar{F}_{06}$$

$$= -Gm\sum_{i=1}^{i=6}\frac{m_1}{|\bar{r}_i|^3}\bar{r}_i$$

GRAVITATIONAL FIELD

- The existence of a particle/body (or mass) changes the space around the particle/body.
- The gravitational field of a particle or body (or anything with mass) is the changed space surrounding that particle or body.
- Any other mass introduced into this gravitational field will feel a gravitational force as a result of its interaction with it.
- The field idea is very useful when dealing with noncontact forces (also known as action at a distance).
- We know that the net gravitational pull on a particle can be caused by another particle or by a distribution of many particles. The gravitational field notion has the advantage of allowing us to measure the net gravitational force on a mass without worrying about whether the force is exerted by a single particle or a spread of particles.
- Every point in a gravitational field has two qualities, one of which is a vector quantity and the other a scalar property. Gravitational Field Intensity is the vector quantity, and Gravitational Potential V is the scalar quantity.

GRAVITATIONAL FIELD INTENSITY (E)

- Gravitational field intensity E at a point in a gravitational field is a vector quantity, defined mathematically as

$$\bar{E} = \underset{\Delta m \to 0}{\text{Lt}} \frac{\bar{F}}{\Delta m},$$

where Δm is an infinitesimally small mass (but not zero) placed at the point, where it experiences a gravitational force **F**. The gravitational field is defined as the gravitational force exerted on a unit mass placed at that spot to obtain a practical estimate. The SI unit of gravitational field intensity is newton per kilogram (N/kg) and its dimensional formula is $[M^0LT^{-2}]$ (same as the dimensional formula of acceleration). Gravitational field strength, or simply Gravitational Field. If a particle with mass m is transported to a position with a gravitational field E, the net gravitational force F acting on that particle at that point is given by **F = mE**

Gravitational Field Intensity Due to A Particle or Point Mass

- Consider a particle of mass m, placed at a point O. We want to determine the gravitational field intensity due to M, at a point near it. is the position vector **r** (taking position as the origin).

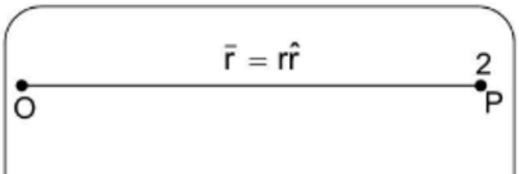


Fig. 3.3

- If a point particle of mass m is placed at P, the gravitational force on it will be GMm/r^2 towards O (i.e., along PO)

$$\bar{F} = -\frac{GMm}{r^2}\hat{r}$$

- The gravitational field intensity at P at the location of the point mass is given by

$$\bar{E} = \frac{\bar{F}}{m} = -\frac{GM}{r^2}\hat{r}$$

- This is defined for all points, wherever the point mass m is kept except at the location of the particle of mass M.

Gravitational Field Intensity Due to A Thin Uniform Ring at A Point on Its Axis

- Consider a thin, uniform ring of mass m and radius a with centre at O. The gravitational field intensity E (due to this ring at a point on the axis of the ring, distant 'r' from the centre of the ring is to be determined.

  Let λ = mass per unit length of ring

$$= \frac{\text{Mass}}{\text{Circumference}}$$

$$= \frac{M}{2\pi a}$$

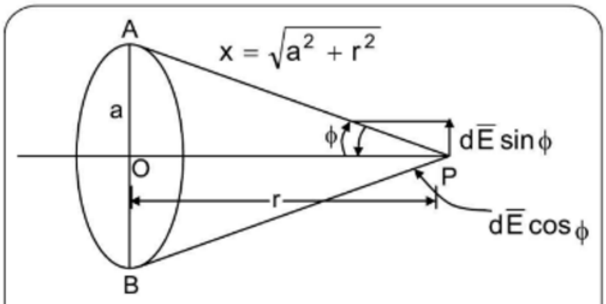


Fig. 3.4

Consider an element of the ring, of length 'dl' at A. Its mass = $dm = \lambda d\ell = \frac{Md\ell}{2\pi a}$

- The gravitational field intensity at P due to this elemental ring at A is

$$d\bar{E} = \frac{Gdm}{x^2},$$

along PA

AP makes an angle ϕ with OP. Now dE can be resolved as dE sinϕ perpendicular to the axis and dE cosϕ along the axis as shown.

dE sinϕ component gets cancelled by the field of a diametrically opposite element at B. Hence the effective component of all ring elements is only dE cosϕ

$$dE\cos\phi = \frac{Gdm}{x^2}\cos\phi$$

$$= \frac{GM}{x^2 \times 2\pi a}\cos\varphi\, d\ell\left(\because dm = \frac{Md\ell}{2\pi a}\right)$$

- The resultant gravitational field at P due to all ring elements is,

$$\bar{E} = \int \overline{dE}\cos\phi, \text{ along PO}$$

$$\therefore \quad E = \int \frac{GM}{x^2 \times 2\pi a}\cos\phi\, d\ell$$

$$= \frac{GM}{2\pi a x^2}\cos\varphi \int_{\ell=0}^{\ell=2\pi a} d\ell = \frac{GM}{x^2}\cos\varphi$$

$$= \frac{GM}{x^2}\cdot\frac{r}{x} \qquad \left(\because \cos\phi = \frac{r}{x}\right)$$

$$= \frac{GMr}{x^3}$$

$$= \frac{GMr}{\left(a^2 + r^2\right)^{\frac{3}{2}}} \qquad \left[\because x = \left(a^2 + x^2\right)^{\frac{1}{2}}\right]$$

$$\therefore \quad E = \frac{GMr}{\left(a^2 + r^2\right)^{\frac{3}{2}}}$$

Notes:

(i) At the centre of the ring (point O), r = 0. Hence E=0 the gravitational field at the centre of the thin, uniform ring is zero.

(ii) If a<<< r

$$E = \frac{GMr}{\left(r^2\right)^{\frac{3}{2}}} = \frac{GM}{r^2}$$

So, for points on the axis which are at a very large distance from the centre of the ring, the ring can be treated as a particle (or point mass).

(iii) The position where the gravitational field E becomes maximum or minimum is determined by putting

$$\frac{dE}{dr}=0$$

$$E=\frac{GMr}{\left(a^2+r^2\right)^{\frac{3}{2}}}$$

$$\frac{dE}{dr}=0 \Rightarrow r=\pm\frac{a}{\sqrt{2}}$$

At $+\frac{a}{\sqrt{2}}$, E is maximum negative (minimum)

At $-\frac{a}{\sqrt{2}}$, E is maximum positive (maximum).

The variation of, along the axis of the ring on either side of the ring is as shown below in Fig.3.4.

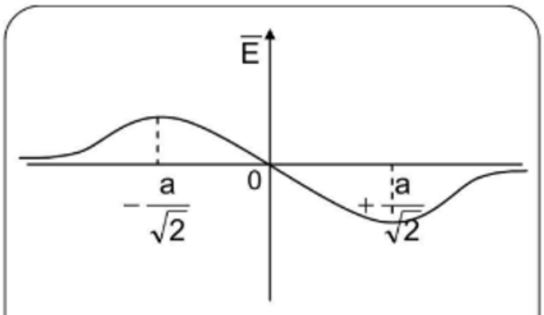


Fig. 3.5

Gravitational Field Intensity Due to A Uniform Disc at A Point on Its Axis

- Consider a uniform disc of mass M and radius a with centre at O. The point is on the axis of the disc at a distance from centre.

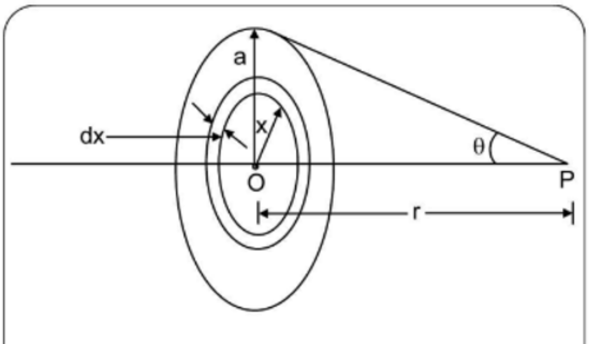


Fig. 3.6

- The disc can be dividend into thin, uniform rings. Consider one such ring of radius x and width along the disc equal to dx

Mass of ring dm = $\frac{\text{Mass of disc}}{\text{Surface Area of disc}}$ × Surface area of ring

$$=\left(\frac{M}{\pi a^2}\right)(2\pi x dx)=\frac{2M}{a^2}xdx$$

- The gravitational field at P due to this elemental ring is

$$dE=\frac{G\,dm\,r}{\left(x^2+r^2\right)^{3/2}} \text{ along PO}$$

$$dE=\frac{Gr}{\left(x^2+r^2\right)^{3/2}}\cdot\frac{2M}{a^2}xdx=\frac{2GMr}{a^2}\cdot\frac{x}{\left(x^2+r^2\right)^{3/2}}dx$$

Gravitational field at P due to disc,

$$E=\int dE=\frac{2GMr}{a^2}\int_{x=0}^{x=a}\frac{xdx}{\left(x^2+r^2\right)^{3/2}}$$

$$=\frac{2GMr}{a^2}\left|-\frac{1}{\sqrt{x^2+r^2}}\right|_0^a=\frac{2GMr}{a^2}\left[\frac{1}{r}-\frac{1}{\sqrt{a^2+r^2}}\right]$$

$$=\frac{2GM}{a^2}\left[1-\frac{r}{\sqrt{a^2+r^2}}\right]$$

$$=\frac{2GM}{a^2}[1-\cos\theta], \text{ where } \cos\theta=\frac{r}{\sqrt{a^2+r^2}}$$

$$E=\frac{2GM}{a^2}[1-\cos\theta]$$

Notes:

(1) When P is very near to centre O θ=90^0, cos θ=0 then

$$\Rightarrow E=\frac{2GM}{a^2} \text{ (Maximum value)}$$

When P is far away from O θ = 0^0, cos θ = 1

E = 0

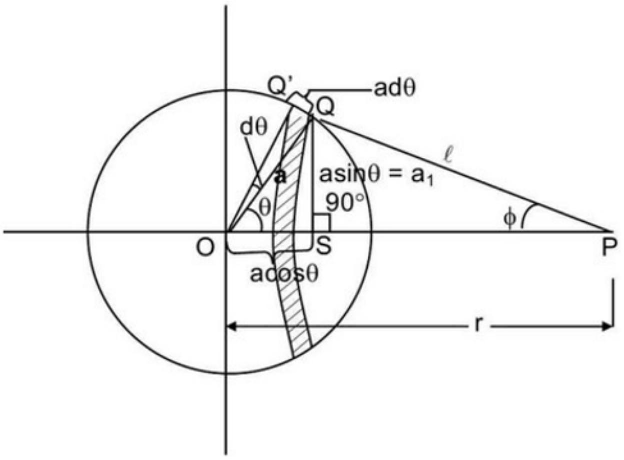
(2) If the disc is infinitely large cos θ = cos90^0 = 0 and

$$E=\frac{2GM}{a^2}$$

for all points on the axis.

Hence the gravitational field due to an infinitely large disc along its axis is uniform (i.e., it is independent of the distance from the disc).

Gravitational Field Due to A Thin, Uniform Shell (Hollow Sphere)



Consider a thin, uniform spherical shell of mass M and radius a, with centre at point O.

Calculation of the gravitational field due to this shell at a point A, distance r from O. Figure 3.7 shows a spherical shell. The shaded area represents a thin ring of radius a1 = asin θ and width Q'Q = adθ and PQ = l angle OPQ = ϕ is ΔPSQ, we have $PQ^2 = PS^2 + QS^2$

$\ell^2 = [r - OS]^2 + QS^2$

$= r^2 - 2r(OS) + (OS)^2 + QS^2$

$\ell^2 = r^2 + a^2 - 2r(OS) \quad [\because QS^2 + OS^2 = OQ^2 = a^2]$

$= r^2 + a^2 - 2ra\cos\theta \quad (\because OS = a\cos\theta)$

$\ell^2 = a^2 + r^2 - 2ar\cos\theta$ ...(i)

area of shaded ring =circumference *width

$= 2\pi a_1 \times a d\theta$

$= 2\pi a\sin\theta . a d\theta$

$= 2\pi a^2 \sin\theta d\theta$

Mass of shaded ring,

$$dm = \frac{M}{\text{Area of shell}} \times \text{area of ring}$$

$$= \frac{M}{4\pi a^2} \times 2\pi a^2 \sin\theta d\theta$$

$$= \frac{M}{2}\sin\theta d\theta \quad \text{...(ii)}$$

The gravitational field at P due to this ring is

$$dE = \frac{Gdm}{\ell^2}\cos\phi$$

(sinϕ components of the ring get cancelled for diametrically opposite points)

$$\therefore dE = \frac{GM}{2}\frac{\sin\theta d\theta \cos\phi}{\ell^2} \quad \text{...(iii)}$$

From ΔPOQ, we have

$a^2 = \ell^2 + r^2 - 2\ell r\cos\phi$

$$\cos\phi = \frac{\ell^2 + r^2 - a^2}{2\ell r} \quad \text{...(iv)}$$

We have

$\ell^2 = a^2 + r^2 - 2ar\cos\theta$ from (i)

Differentiating (i), we get

$2\ell d\ell = 2ar\sin\theta d\theta$

$$\therefore \quad \sin\theta d\theta = \frac{\ell d\ell}{ar} \quad \text{...(v)}$$

while solving we get

$$dE = \frac{Gm}{4ar^2}\left[1 - \frac{(a^2 - r^2)}{\ell^2}\right]d\ell$$

$$E = \int dE = \int_{\ell_1}^{\ell_2} \frac{Gm}{4ar^2}\left[1 - \frac{(a^2 - r^2)}{\ell^2}\right]d\ell$$

$$E = \frac{Gm}{4ar^2}\left[\ell + \frac{(a^2 - r^2)}{\ell}\right]_{\ell_1}^{\ell_2}$$

The following cases are of particular interest:

(i) P outside the shell (r>a)

In this case, value of l varies from (r-a) to (r+a)

$$\therefore \quad E = \frac{Gm}{4ar^2}\left[\ell + \frac{(a^2 - r^2)}{\ell}\right]_{r-a}^{r+a}$$

$$= \frac{Gm}{4ar^2}\left[(r+a) + (a-r) - \{(r-a) - (a+r)\}\right]$$

$$= \frac{Gm}{4ar^2}[2a + 2a] = \frac{Gm}{r^2}$$

for all outside points $E = Gm/r^2$ As a result, for the computation of the gravitational field at the external points of a thin, uniform spherical shell, the shell can be considered as a point mass (particle) kept at the shell's geometric centre.

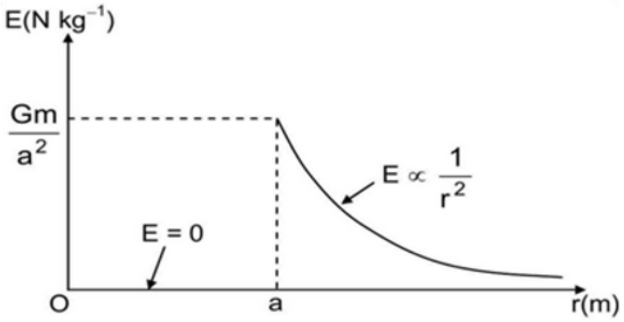
(ii) inside the shell (r<a)

In this case, l varies from (a-r) to (a+r)

$$\therefore\ E=\frac{Gm}{4ar^2}\left[\ell+\frac{(a^2-r^2)}{\ell}\right]_{a-r}^{a+r}$$

$$=\frac{Gm}{4ar^2}\left[(a+r)+(a-r)-\{(a-r)+(a+r)\}\right]$$

E=0 for inside points of shell. Hence, the gravitational field inside a thin, uniform spherical shell due to the mass of the shell, is zero.

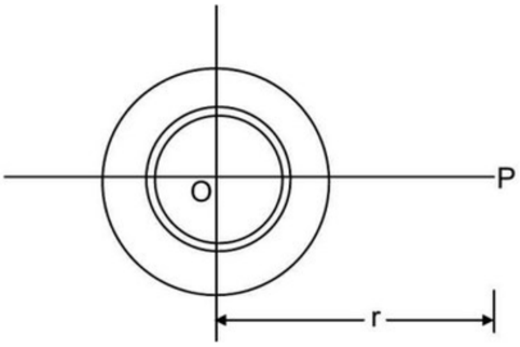


Variation of gravitational field due to uniform, thin spherical shell of radius ' a.'

Gravitational Field Intensity Due to A Uniform Solid Sphere

- Consider a uniform solid sphere of mass m and radius 'a' with centre at point O. Let us evaluate its gravitational field intensity at a point P, distant ' r ' from O.

(i) P is outside the sphere (r>a)



The solid sphere can be divided into concentric uniform thin shells, each of mass dm The gravitational field at P due to thin shell is

$$dE=\frac{Gdm}{r^2}$$

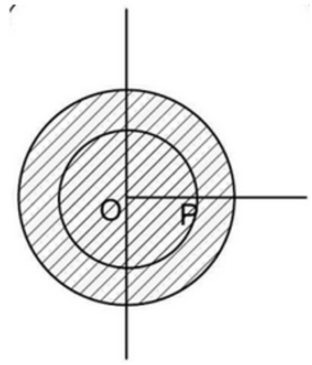
Total gravitational field at P due to solid sphere

$$E=\int dE=\int\frac{Gdm}{r^2}=\frac{GM}{r^2}$$

$E=\frac{GM}{r^2}$ for outside points

- As a result, a uniform solid sphere can be considered as a point mass (particle) held at its geometric centre for calculating the gravitational field at all places outside the solid sphere.

(ii) P is inside the sphere (r<a)



In this case, we can treat the solid sphere as made of two parts, namely, (1) solid sphere of radius ' r ' and (2) uniform spherical shell of inside radius and outside radius a. The gravitational field at P is due to the superposition of the gravitational fields due to these two portions. We know that gravitational field at due to the shell is zero (as P is in the inside of the shell). Hence gravitational field intensity at P is due to a solid sphere of radius (instead of a).

Mass of reduced sphere,

$$M'=\frac{M}{\left(\frac{4}{3}\pi a^3\right)}\times\frac{4}{3}\pi r^3=\frac{M}{a^3}r^3$$

gravitational field at P is E

$$E=\frac{GM'}{r^2}$$

$$=\frac{G}{r^2}\frac{M}{a^3}r^3=\frac{GMr}{a^3}$$

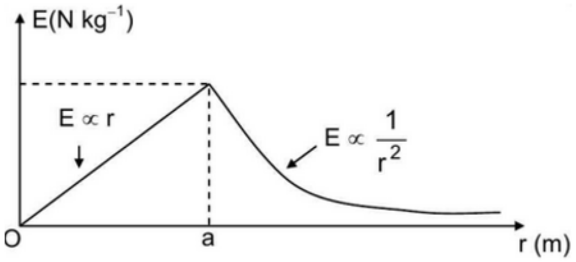
$$E=\left(\frac{GM}{a^3}\right)r$$

for inside points of solid sphere i.e.,

$E\propto r$

for inside points, Hence the gravitational field intensity at p inside points of a uniform solid sphere, is directly proportional to the distance of that point from the geometric centre of the solid sphere.

Since r = 0 at the centre of the solid sphere, the gravitational field intensity at the centre of the solid sphere is zero.



Variation of gravitational field intensity due to a uniform solid sphere of radius 'a' is shown in Fig.3.11.

GRAVITATIONAL POTENTIAL (V)

- The gravitational potential at a place is equal to the work done by an external force to transport a particle of unit mass from infinity to its position in the gravitational field. Gravitational potential is a scalar quantity. Its SI unit is joule per kilogramme (Jkg^{-1}) and its dimensional formula is $[M^0L^2T^{-2}]$.
- While bringing the unit mass from infinity to its position in the gravitational field, the applied external force is equal and opposite to the gravitational force on the particle at those points, i.e., the particle is slowly brought from infinity to its position so that its kinetic energy is zero at all positions.
- Hence gravitational potential V can also be defined as the negative of the work done by the gravitational force as a particle of unit mass is brought from infinity to its position in the gravitational field.
- If ' W ' is the work done by an external force, in bringing a particle of mass ' m ' from infinity to its position in the gravitational field, then the gravitational potential at that point (V) is given by

$$V = \frac{W}{m}$$

- Also, $W = -W_G$, where W_G work done by gravitational force in bringing the particle from infinity to its position

$$W_G = \int_{\infty}^{r} \overline{F}_G . \overline{dr}$$

where F_G gravitational force on particle and dr = displacement of particle

$$\Rightarrow W = -\int_{\infty}^{r} \overline{F}_G . \overline{dr} \quad \therefore V = \frac{W}{m} = -\frac{W_G}{m} = \frac{-\int_{\infty}^{r} \overline{F}_G . \overline{dr}}{m}$$

$$= -\int_{\infty}^{r} \frac{\overline{F}_G}{m} . \overline{dr}$$

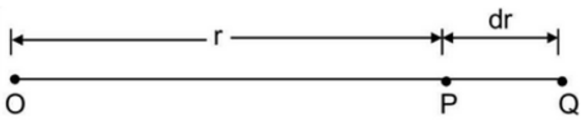
$$\therefore V = -\int_{\infty}^{r} \overline{E} . \overline{dr}$$

$$\left(\because \frac{\overline{F}_G}{m} = \overline{E} \right)$$

Conventionally the potential of a particle at infinity is taken as zero.

Gravitational Potential V At a Distance r From a Point Mass (M)

- Consider a particle of mass M placed at point O. We want to determine the gravitational potential at point P, distance r from O.



- The gravitational force acting on a particle of unit mass at P is the gravitational field intensity at P due to the mass at O.

$F_G = E = \frac{GM}{r^2}$, along PO.

- If the unit mass is displaced from P through a small distance dr to Q, the small amount of work done by gravitational force

$$dW_G = \overline{F}_G . \overline{dr} = \overline{E} . \overline{dr}$$

= E dr cos 180°

($\because$ $\overline{E}$ and $\overline{dr}$ are in opposite directions)

$$= -E\,dr = -\frac{GM}{r^2} dr$$

- Work done by the gravitational force in transferring the unit mass from infinity to P is given by

$$W_G = \int_{\infty}^{r} dW_G = \int_{\infty}^{r} -\frac{GM}{r^2} dr$$

Gravitational potential at P is given by

$$V = -W_G = -\int_{\infty}^{r} -\frac{GM}{r^2} dr = -GM\left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$= -\frac{GM}{r} + \frac{GM}{\infty} = -\frac{GM}{r}$$

$$V = -\frac{GM}{r}$$

- Since gravitational potential at infinite distance is considered to be zero then the gravitational potential comes out to be always negative

<u>Relation between gravitational field intensity (E) and gravitational potential (V)</u>

Since V is obtained by integration of E, the converse, namely differentiation of V gives E. Since V is in general V=V(x,y,z)

$\bar{E} = -\bar{\nabla}V$, where

$$\bar{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}, \text{ i.e.,}$$

$$\bar{E} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right]$$

is the relation between gravitational field intensity E and gravitational potential V

When V depends on x alone,

$$E_x = -\frac{dV}{dx}, \text{ where } E = \left|\bar{E}\right|$$

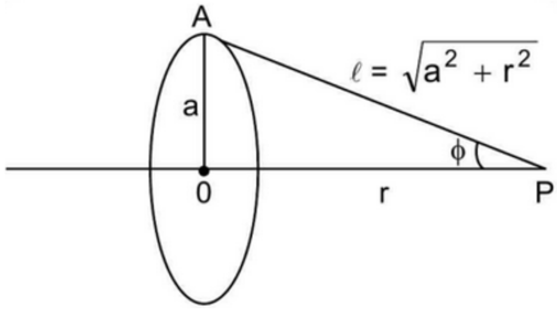
Similarly, $E_y = \frac{-dV}{dy}$ or $E_z = -\frac{dV}{dz}$

we can also write,

$E = -\frac{dV}{dr}$, when V is spherically symmetric

Gravitational Potential Due to A Thin Uniform Ring Along the Axis of The Ring

- Consider a thin, uniform ring of mass m and radius a with centre at O.



- An element of length dl of the ring at A has a mass dm

$$= \frac{Md\ell}{2\pi a}$$

- The gravitational potential at P on the axis of the ring, distance r from the centre O, due to dm is given by

$$dV = \frac{-G\,dm}{\ell}$$

Total gravitational potential at p due to the ring

$$V = \int dV = \int_0^M -\frac{Gdm}{\ell} = -\frac{GM}{\ell}$$

$$\therefore V = \frac{-GM}{\left(a^2 + r^2\right)^{1/2}}$$

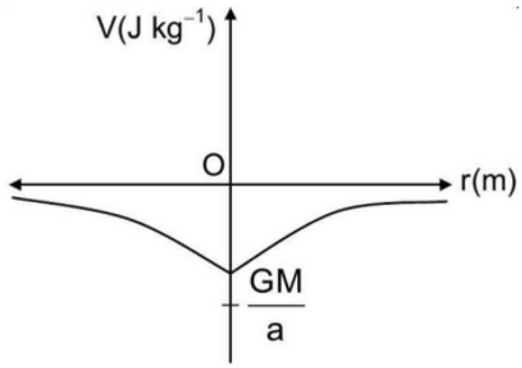
At the centre of the ring, r = 0

$$\therefore V = -\frac{GM}{a}$$

If r>>a

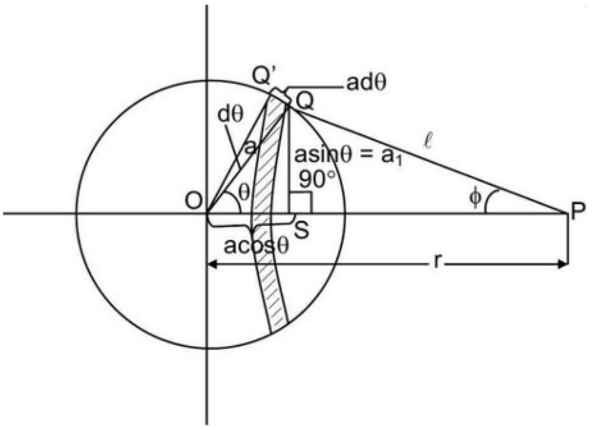
$$V = -\frac{GM}{r}$$

i.e., for distant points along the axis, the thin uniform ring behaves like a particle of mass m at its centre.



- Variation of gravitational potential along the axis of a thin, uniform ring of radius 'a' is given in figure.

Gravitational Potential Due to A Thin Uniform Spherical Shell



- Consider a thin, uniform shell of mass M and radius a with centre at point O. The mass of the element ring (shaded area) is

  $dm = \frac{M}{2}\sin\theta d\theta$ (already derived in the section on gravitational field)

  Also (from Fig. 3.15)

  $\ell^2 = a^2 + r^2 - 2ar\cos\theta$

  $\therefore\ 2\ell d\ell = 2ar\sin\theta d\theta$

  $\Rightarrow\ \sin\theta d\theta = \frac{\ell d\ell}{ar}$

  $\therefore dm = \frac{M\ell d\ell}{2ar}$ (This is also derived in the section on gravitational field)

- The gravitational potential at P due to dm is given by

  $dV = -\frac{G\,dm}{\ell}$ (for ring)

  $= -\frac{GM\ell d\ell}{2ar\ell}$

  $dV = -\frac{GMd\ell}{2ar}$

  As we vary θ from zero to π, the rings formed on the shell cover up the whole shell. The potential due to the shell is obtained by integrating dV within the limits θ = 0 to π

(i) P outside the shell (r > a)

$\ell^2 = a^2 + r^2 - 2ar\cos\theta$

when θ=0, l = r – a and when θ = π, l = r + a

$$\therefore\quad V = \int dV = \int_{r=(r-a)}^{r=(r+a)} -\frac{GM}{2ar}d\ell$$

$$= -\frac{GM}{2ar}\left|\ell\right|_{r-a}^{r+a}$$

$$= -\frac{GM}{2ar}\left[(r+a)-(r-a)\right] = -\frac{GM}{r}$$

$\therefore\quad V = -\frac{GM}{r}$ for all external points

As a result, the thin uniform shell can be treated as a point mass with the same mass as the shell, positioned in the shell's centre for the calculation of gravitational potential at all external points.

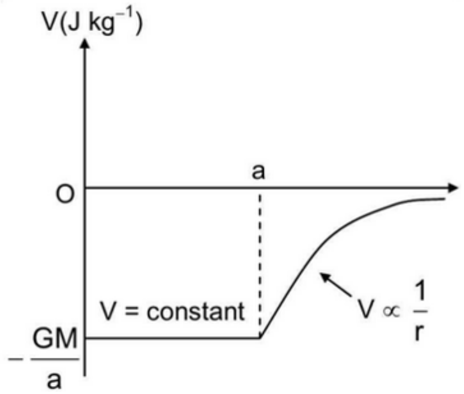
(ii) P inside the shell (r < a)

In this case, when θ = 0, l = a – r and when θ = π, l = a + r

$$V = \int dV = \int_{a-r}^{a+r} -\frac{GM}{2ar}d\ell = -\frac{GM}{a}$$

$\therefore\ V = -\frac{GM}{a}$ (Inside the shell, V is independent of 'r')

As a result, the gravitational potential of a thin, homogeneous spherical shell is the same within and outside of it. In other words, the gravitational field is uniform. As a result, an equipotential volume is the interior of a thin, uniform spherical shell.



Variation of gravitational potential due to a thin, uniform spherical shell of radius ' ' is shown in Fig. 3.16.

We know that E = 0, inside a spherical shell due to its mass alone. Also

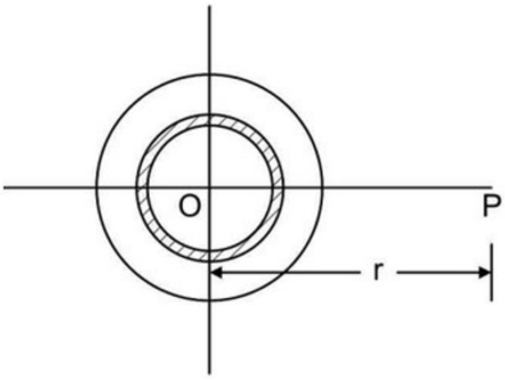
$$E = -\frac{dV}{dr} \Rightarrow \frac{dV}{dr} = 0 \Rightarrow V = \text{constant}$$

inside a thin, uniform spherical shell. Thus, it is not necessary that if the gravitational field is zero at a point, the gravitational potential is zero at that point.

Gravitational Potential Due to A Uniform Solid Sphere

(i) P outside the sphere(r>a)

Consider a uniform solid sphere of mass M and radius a, with centre at O.



The solid sphere can be divided into a large number of concentric uniform spherical shells, each of mass. The gravitational potential at P due to the shell of mass dm is

$$dV = -\frac{G\,dm}{r}$$

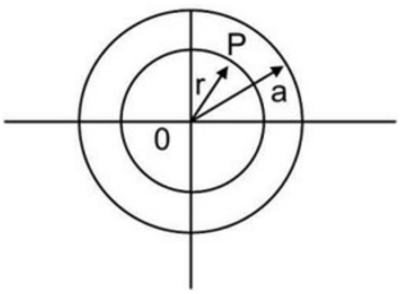
Total gravitational potential at P due to the entire sphere,

$$V = \int dV = \int_0^M -\frac{Gdm}{r} = -\frac{GM}{r}$$

$\therefore\ V = -\frac{GM}{r}$ for all external points

Hence a solid uniform sphere can be treated as a particle at its centre, having the same mass as the sphere, for calculation of gravitational potential at all external points.

(ii) P inside the sphere (r<a)



The gravitational potential V at P is due to

(a) a uniform solid sphere of radius r and mass M_s and

(b) a hollow sphere of outside radius a and inside radius r

$V = V_s + V_H$

V_s = potential due to solid sphere of mass M_s and

V_H = potential due to hollow sphere

$$M_S = \frac{M}{\left(\frac{4}{3}\pi a^3\right)} \times \frac{4}{3}\pi r^3 = \frac{Mr^3}{a^3}$$

$$\therefore\ V_S = -\frac{GM_S}{r} = \frac{GMr^3}{a^3 r} = \frac{-GMr^2}{a^3} \quad \text{....(i)}$$

For calculation of V_H, we take an elemental shell of radius x and thickness dx ($x_{min} = r$, $x_{max} = a$)

$$dM_H = \frac{M}{\left(\frac{4}{3}\pi a^3\right)} \cdot 4\pi x^2 dx = \frac{3Mx^2 dx}{a^3}$$

$$dV_H = -\frac{GdM_H}{x} = \frac{-3GMxdx}{a^3}$$

$$\therefore\ V_H = \int dV_H = \int_{x=t}^{x=a} \frac{-3GMxdx}{a^3} = -\frac{3GM}{a^3}\left|\frac{x^2}{2}\right|_r^a$$

$$= -\frac{3GM}{2a^3}\left(a^2 - r^2\right)$$

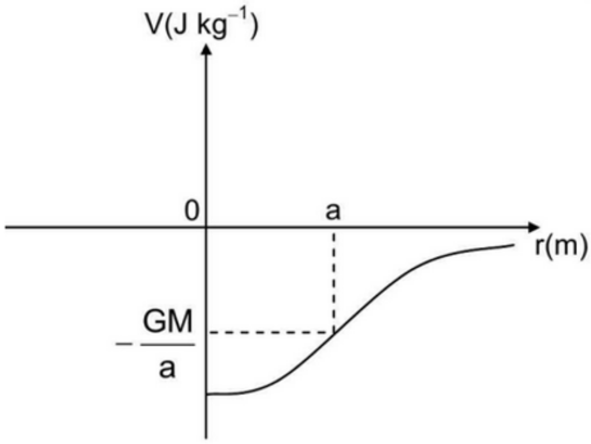
$$\therefore\ V = V_S + V_H = -\frac{GMr^2}{a^3} - \frac{3GM}{2a^3}\left(a^2 - r^2\right)$$

$$= -\frac{GM}{2a^3}\left[2r^2 + 3a^2 - 3r^2\right] = -\frac{GM}{2a^3}\left[3a^2 - r^2\right]$$

$$\therefore\ V = -\frac{GM}{2a^3}\left(3a^2 - r^2\right)$$

for interior points. At the centre of the sphere, r=0

$$\Rightarrow\ V = -\frac{3}{2}\frac{GM}{a}$$



Variation of gravitational potential due to a uniform solid sphere of radius 'a' is shown in figure.

GRAVITATIONAL POTENTIAL ENERGY

- Because gravitational force is a conservative force, gravitational potential energy can be defined as the energy associated with a system of particles interacting via gravitational force.
- We had earlier defined the gravitational potential $V = W_G/m$, W_G = work done by gravitational force on a particle of mass m, in bringing it slowly from infinity to its position in the gravitational field.
- Since $-W_G$ is the negative of the work done by a conservative force, it is equal to the change in potential energy between the final and initial positions ($\Delta U = -W_G$, where ΔU equals the change in potential energy)

  $$\therefore\ V = \frac{\Delta U}{m} = \frac{U(r) - U(\infty)}{m}$$

  U(r) = gravitational potential energy of the particle at position

  U() = gravitational potential energy at infinity (conventionally taken as zero)

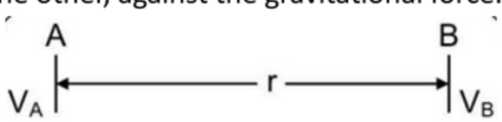
  $$V = \frac{U(r)}{m}$$

- That is, the gravitational potential energy of a unit mass placed at a point in a gravitational field is the gravitational potential energy of that point.

  U = mV is the gravitational potential energy of a particle of mass m, placed at a point in a gravitational field, where the gravitational potential is V. Gravitational potential energy of a particle is zero at infinite distance or it is always negative.

GRAVITATIONAL POTENTIAL DIFFERENCE

- The gravitational potential difference between two points in a gravitational field is the change in potential energy of a particle of unit mass, when it is moved from one point to the other, against the gravitational force.



Let A and B be two points, at gravitational potentials V_A and V_B respectively. If a particle of mass m is placed at A, its potential energy at A is $U(A) = mV_A$ .... (i)

If this particle is moved very slowly from A to B, at B, its potential energy is $U(B) = mV_B$ .... (ii) ΔU = change in potential energy = U(B) – U(A) = $mV_B - mV_A$

$= m(V_B - V_A)$

potential difference between and is given by

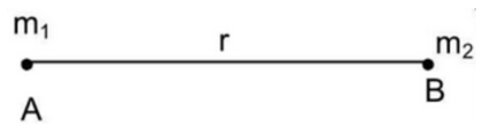
$$\Delta V = \frac{\Delta U}{m} = \frac{m(V_B - V_A)}{m} = V_B - V_A$$

If A is at infinity, $V_A = 0$, $\Delta V = V_B$

i.e., At any point in a gravitational field, gravitational potential is the change in potential energy of a particle of unit mass brought from infinity to that point in the gravitational field.

Gravitational Potential Energy of a System of Two Particles

- The gravitational potential energy of a two-particle system is the inverse of the gravitational force's effort in constructing the system by bringing the particles from infinity to the required configuration.
- Consider two particles of masses m_1 and m_2, placed at A and B respectively, separated by a distance ' r '.



- If we consider that these particles were initially at infinite distance, initial potential energy of A $m_1 V_\infty = 0$:

  and initial potential energy of B

  $= m_2 V_\infty = 0$ ($\because V_\infty = 0$)

- If particle m_1 was brought from infinity to A, no work is done as there is no gravitational field (or force). But m_1 sets up a gravitational field all around A so that gravitational potential at B is V_B

  Potential energy of m_2 when it is brought to B is = $m_2 V_B = -\frac{Gm_1 m_2}{r}$