Toppersnotes®
Unleash the topper in you

हरियाणा

P.G.T.

हरियाणा लोक सेवा आयोग (HPSC)

Mathematics

Volume - 3

SIERRA INNOVATIONS

विषय सूची

Toppersnotes
Unleash the topper in you

सदिश कलन

यदि सदिश $\vec{r}$, अदिश t का फलन हो।

Q.) यदि $\vec{r} = 3t^2\hat{i} + (t^3+1)\hat{j} + 4t\hat{k}$ हो then

i) $\frac{d\vec{r}}{dt} = 6t\hat{i} + 3t^2\hat{j} + 4\hat{k}$

ii) $\left|\frac{d\vec{r}}{dt}\right|_{t=1 \text{ पर}} = 6\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \left|\frac{d\vec{r}}{dt}\right|_{t=1} = \sqrt{36+9+16} = \sqrt{61}$

iii) $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$

$= 6\hat{i} + 6t\hat{j} + 0\hat{k}$

$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds}$

वक्र $\vec{r} = f(t)$ के किसी बिन्दु पर अवकलज $\frac{d\vec{r}}{dt}$ की दिशा, $d\vec{s}$ के उस बिन्दु पर स्पर्शी की दिशा में होती है।

$\therefore$ वक्र $\vec{r} = f(t)$ के point $P(t)$ पर स्पर्शी की दिशा में इकाई सदिश $\frac{\left(\frac{d\vec{r}}{dt}\right)}{\left|\frac{d\vec{r}}{dt}\right|}$

Q.) यदि $\vec{r} = (x^2yz)\hat{i} + (xy^2z)\hat{j} + (xyz^2)\hat{k}$ हो

then (i) $\frac{\partial\vec{r}}{\partial x} = (2xyz)\hat{i} + (y^2z)\hat{j} + (yz^2)\hat{k}$

(ii) $\frac{\partial\vec{r}}{\partial z} = (x^2y)\hat{i} + (xy^2)\hat{j} + (2xy)\hat{k}$

(iii) $\frac{\partial^2 \vec{r}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left[(x^2 z)\hat{i} + (2xyz)\hat{j} + (xz^2)\hat{k}\right]$

$= (2xz)\hat{i} + (2yz)\hat{j} + (z^2)\hat{k}$

(iv) $\frac{\partial^2 r}{\partial x^2} = (2yz)\hat{i} + 0 + 0$

(v) $\frac{\partial^2 r}{\partial y^2} = (2xz)\hat{j}$

Q.42) $\vec{r} = (\cos nt)\hat{i} + (\sin nt)\hat{j}$

$\vec{r} \times \frac{d\vec{r}}{dt} = \left[(\cos nt)\hat{i} + \sin nt\,\hat{j}\right] \times \left[-n\sin nt\,\hat{i} + n\cos nt\,\hat{j}\right]$

$\because \hat{i}\times\hat{i} = 0$
$\hat{j}\times\hat{j} = 0$

$= n\cos^2 nt\,\hat{k} + n\sin^2 nt\,\hat{k}$

$= n(1)\hat{k} = n\hat{k}$

Result –

① $\frac{d}{dt}(\vec{a})^2 = 2\vec{a}\frac{d\vec{a}}{dt}$

prove

$\because (\vec{a})^2 = \vec{a}\cdot\vec{a}$

L.H.S. $\frac{d}{dt}(\vec{a}\cdot\vec{a}) = \vec{a}\cdot\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{a}$

$= \vec{a}\cdot\frac{d\vec{a}}{dt} + \vec{a}\frac{d\vec{a}}{dt}$

$= 2\vec{a}\frac{d\vec{a}}{dt}$

② $\vec{a}\cdot\frac{d\vec{a}}{dt} = a\left(\frac{da}{dt}\right)$ —①

prove –

$\because (\vec{a})^2 = \vec{a}\cdot\vec{a}$
$= a\,a\cos 0$
$= a^2$

$\therefore \frac{d}{dt}(\vec{a})^2 = \frac{d}{dt}(a^2)$

$= 2a\frac{da}{dt}$ —②

by ① & ② $\Rightarrow \boxed{\vec{a}\cdot\frac{d\vec{a}}{dt} = a\left(\frac{da}{dt}\right)}$

Q) ~~यदि फलन~~ $f(x,y,z) = x^2 y^3 z^4$ ~~एक अदिश फलन है~~
then ∇f ज्ञात करो।

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 y^3 z^4)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2 y^3 z^4) + \hat{j}\frac{\partial}{\partial y}(x^2 y^3 z^4) + \hat{k}\frac{\partial}{\partial z}(x^2 y^3 z^4)$$

$$\boxed{\nabla f = 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}}$$

Q. यदि $f(x,y,z) = x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}$
हो then (i) ∇f व (ii) $\nabla \times f$ ज्ञात करो।

Sol.^{n} (i) $\nabla \cdot f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}\right)$

$\hat{i} \cdot \hat{i} = 1$

$$= 2xyz + 2xyz + 2xyz$$

$$= 6xyz$$

(ii) $\nabla \times f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}\right)$

$\hat{i} \times \hat{j} \times \hat{k}$

$\hat{i} \times \hat{i} = 0$

$$= y^2z\hat{k} + yz^2\hat{j} + (-x^2z\hat{k}) -$$

$$\nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \hat{i}(xz^2 - xy^2) - \hat{j}(yz^2 - x^2y) + \hat{k}(y^2z - x^2z)$$

$$\nabla \times f = x(z^2 - y^2)\hat{i} - y(z^2 - x^2)\hat{j} + z(y^2 - x^2)\hat{k}$$

प्रवणता पर प्रमेय ⇒

(v) अदिश फलन $\phi(x,y,z)$ की प्रवणता $\nabla\phi$ एक सदिश है।

prove let एक अदिश फलन $\phi(x,y,z) = c$ है।

$$\therefore \quad d\phi(x) = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\phi\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) = 0$$

$$\Rightarrow (\nabla\phi)\cdot(d\vec{r}) = 0 \qquad (\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k})$$

⇒ ~~∇ के~~ $\nabla\phi$, सदिश $d\vec{r}$ के $\perp$ है।

⇒ $\nabla\phi$, सतत $\phi(x,y,z) = c$ के बिन्दु $p(\vec{r})$ पर अभिलम्ब की दिशा में सदिश है।

समतल का समी॰

Q.12) let given surface $\phi : xyz - 4 = 0$ है।

$$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xyz - 4)$$

$$= yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$\therefore$ point $(1,2,2)$ पर-

$$\nabla\phi = 4\hat{i} + 2\hat{j} + 2\hat{k}$$

$\therefore$ दी सतह के बिन्दु $(1,2,2)$ पर स्पर्श तल का

समी॰ ⇒ $4(x-1) + 2(y-2) + 2(z-2)$

$$\Rightarrow \boxed{4x + 2y + 2z = 12}$$

Q.13) point $p(1,2,2)$ पर अभिलम्ब का समी॰

$$\frac{x-1}{4} = \frac{y-2}{2} = \frac{z-2}{2}$$

या $\boxed{\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}}$

प्रवणता पर आधारित परिणाम –

① $grad\ r = \nabla r = \hat{r}$

prove
$$\nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(r)$$
$$= \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$
$$= \left(\frac{x}{r}\right)\hat{i} + \left(\frac{y}{r}\right)\hat{j} + \left(\frac{z}{r}\right)\hat{k}$$
$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r} = \hat{r}$$

② $\nabla f(r) = grade \cdot f(r) = f'(r)\nabla r$
$$\nabla f(r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(r)$$
$$= \hat{i} f'(r)\frac{\partial r}{\partial x} + \hat{j} f'(r)\frac{\partial r}{\partial y} + \hat{k} f'(r)\frac{\partial r}{\partial z}$$
$$= f'(r)\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$$
$$= f'(r)\,\hat{r}$$

या $\boxed{\nabla f(r) = f'(r)\,\nabla r}$

Special case – जब $f(r) = \frac{1}{r}$ हो तब

① $\nabla\left(\frac{1}{r}\right) = \left(-\frac{1}{r^2}\right)\cdot\hat{r}$
$$= \left(-\frac{1}{r^3}\right) r\,\hat{r}$$
$$\boxed{\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^2}}$$

② $\boxed{\nabla(e^r) = e^r(\hat{r})}$

③ $\boxed{\nabla(\log r) = \frac{1}{r}\cdot\hat{r}}$

(4) $\nabla(r^m) = m r^{m-1}\cdot\hat{r}$
$= m r^{m-1}\nabla r$

Q.47) $\nabla e^{r^2} = e^{r^2}\cdot\hat{r}$

आगे है→ $= 2r\cdot e^{r^2}\cdot\hat{r}$
$= 2e^{r^2}\cancel{r}\cdot\frac{\vec{r}}{\cancel{r}}$
$= 2e^{r^2}\cdot\vec{r}$

Q.1) $\vec{r} = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$

$\left|\frac{d\vec{r}}{dt}\right| = \cos t\hat{i} - \sin t\,\hat{j} + \hat{k}$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t + 1}$

$= \sqrt{1+1} = \sqrt{2}$

Q.2) $\vec{r} = 3\hat{i} - 6t^2\hat{j} + 4t\hat{k}$

$\frac{d\vec{r}}{dt} = -12t\,\hat{j} + 4\hat{k}$

$\frac{d^2\vec{r}}{dt^2} = -12\hat{j} + 0 = -12\hat{j}$

Q.3) किसी सदिश $f(t)$ का परिमाण अचर होगा iff $F\cdot\frac{dF}{dt} = 0$

Q.4) $\vec{a} = 5t^2\hat{i} - \cos t\,\hat{j}$
$\vec{b} = t\hat{i} - \sin t\,\hat{j}$

$(\vec{a}\cdot\vec{b}) = 5t^3 + \sin t\cos t = 5t^3 + \frac{\sin 2t}{2}$

$\frac{d}{dt}(\vec{a}\cdot\vec{b}) = 15t^2 + \frac{\cancel{2}\cos 2t}{\cancel{2}}$

Q.5) $x = 4\cos t$, $y = 4\sin t$, $z = 6t$
वक्र पर किसी चर बिन्दु का स्थिति सदिश →

$\vec{r} = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 6t\hat{k}$

$\vec{v} = \frac{d\vec{r}}{dt} = -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 6\hat{k}$

त्वरण $= \frac{d\vec{v}}{dt} = -4\cos t\,\hat{i} - 4\sin t\,\hat{j}$

$t=1$ पर →

$\left|\frac{d\vec{v}}{dt}\right|_{t=1} = \sqrt{}$

Q.8) $f(x,y,z) = x^3 - y^3 + x^2 z$

$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^3 - y^3 + x^2 z)$

$= 3x^2\hat{i} - 3y^2\hat{j} + x^2\hat{k}$

$(\nabla f)_{(1,-1,2)} = 3\hat{i} - 3\hat{j} + \hat{k}$

Q.9) $f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial z}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

$= \left(\frac{2x}{a^2}\hat{i} + \frac{2y}{b^2}\hat{j} + \frac{2z}{c^2}\hat{k}\right)$

Q.10) $f = xy + yz + zx$

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(xy+yz+zx)$$

$$\nabla f = \left[(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right]$$

point $(3,1,2)$ पर —

$$\nabla f = 3\hat{i} + 5\hat{j} + 4\hat{k} \quad \text{—(1)}$$

Now $2\hat{i}+3\hat{j}+6\hat{k}$ की दिशा में इकाई सदिश —

$$\hat{a} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{2^2+3^2+6^2}}$$

$$\Rightarrow \hat{a} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7} \quad \text{—(2)}$$

अतः अभीष्ट दिक् अवकलज

$$\frac{df}{ds} = \text{grad } f \cdot \hat{a}$$

$$= (3\hat{i}+5\hat{j}+4\hat{k})\cdot\left(\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)$$

$$= \frac{6+15+24}{7} = \frac{45}{7}$$

Q11) $$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2y+2xy-4)$$

$$\nabla f = \left[(2xy+2x)\hat{i} + (x^2+2x)\hat{j}\right] \quad \text{—(1)}$$

∵ अभिलम्ब का सदिश समी.

$$(\vec{r} - \vec{r}_0) \times \nabla f = \vec{0} \quad \text{—(2)}$$

∵ point $(1,-2,3)$ पर —

$$\text{grade } f = [2(1)(-2)+2(1)]\hat{i} + (4+2)\hat{j}$$

$$\nabla f = -2\hat{i} + 6\hat{j} \quad \text{—(3)}$$

$$\vec{r}_0 = \hat{i} - 2\hat{j} + 3\hat{k} \quad \text{—(4)}$$

put in (2) ⇒

$$\Rightarrow \left[(x\hat{i}+y\hat{j}+z\hat{k}) - (\hat{i}-2\hat{j}+3\hat{k})\right] \times (-2\hat{i}+6\hat{j})$$

$$\Rightarrow \left[(x-1)\hat{i}+(y+2)\hat{j}+(z-3)\hat{k}\right] \times (-2\hat{i}+6\hat{j})$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-1 & y+2 & z-3 \\ -2 & 6 & 0 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \hat{i}(0-6z+18) - \hat{j}(0+2z-6) + \hat{k}(6x-6+2y+4)$$

$$\Rightarrow 6(z+3)\hat{i} - 2(z-3)\hat{j} + 2(3x+y-1)\hat{k}$$

Q.11) ∵ given $\phi = x^2y + 2xz - 4$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y+2xz-4)$$

$$= (2xy+2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

at point $(1,-2,3)$ ⇒

$$\nabla\phi = 2\hat{i} + \hat{j} + 2\hat{k}$$

∴ सतह के point $(1,-2,3)$ पर अभिलम्ब एकक सदिश ⇒

$$\hat{n} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}}$$

$$\boxed{\hat{n} = \frac{1}{3}(2\hat{i}+\hat{j}+2\hat{k})} \quad \text{Ans.}$$

Q.14) $f = x\hat{i} + (x+y)\hat{j} + (x+y+z)\hat{k}$

$$\text{div} f = \frac{\partial x}{\partial x} + \frac{\partial}{\partial y}(x+y) + \frac{\partial}{\partial z}(x+y+z)$$

$$= \cancel{\hat{i}+\hat{j}+\hat{k}} \quad 1+1+1$$

$$= 3$$

Q.17) $$\text{div}\, \vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\cancel{x\hat{i}+y\hat{j}+z\hat{k}}\right) \frac{\vec{r}}{r}$$

$$= \left(\frac{\partial \hat{i}}{\partial x} + \frac{\partial \hat{j}}{\partial y} + \frac{\partial \hat{k}}{\partial z}\right)\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum \left[1 \cdot \frac{1}{r} + x\left(\frac{-1}{r^2}\right)\frac{x}{r}\right]$$

$$= \sum \frac{1}{r} - \frac{1}{r^3}\sum x^2$$

$$= \frac{3}{r} - \frac{1}{r^3}(r^2) = \frac{2}{r}$$

Q.18) $f = (xy^2 + yz^3)$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$$

$$\text{grad} f = \hat{i}\, y^2 + (2xy + z^3)\hat{j} + (3yz^2)\hat{k}$$

Now point $(2,-1,1)$ पर —

$$\text{grade}\, \cancel{\phi} = +\hat{i} + (-4+1)\hat{j} + (-3)\hat{k}$$

$$\nabla f = +\hat{i} - 3\hat{j} - 3\hat{k}$$

साथ ही $\hat{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ की दिशा में एकक सदिश $\Rightarrow$

$$= \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}}$$

$$= \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

अतः अभीष्ट दिक्अवकलज

$$\frac{df}{ds} = (\nabla f)\,\hat{a}$$

$$= (\hat{i} - 3\hat{j} - 3\hat{k})\frac{(\hat{i} + 2\hat{j} + 2\hat{k})}{3}$$

$$= \frac{1}{3}(+1 - 6 - 6) = \frac{-11}{3}$$

Q.47) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = |\vec{r}|$ हो तब

$$\nabla(e^{r^2}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(e^{r^2})$$

$$= \sum \left[\hat{i}\frac{\partial}{\partial x}(e^{r^2})\right]$$

$$= \sum \hat{i}\left(e^{r^2} \cdot 2r\frac{\partial r}{\partial x}\right)$$

$$= 2r \cdot e^{r^2} \sum\left(\hat{i}\frac{x}{r}\right)$$

$$= 2 \cdot e^{r^2}\left(\frac{\vec{r}}{r}\right)$$

$$= 2 e^{r^2}(\vec{r})$$

$$\boxed{\begin{aligned}\nabla(e^{r^2}) &= e^{r^2}(2r)\cdot(\nabla r) \\ &= 2r \cdot e^{r^2}\left(\frac{\vec{r}}{r}\right) \\ &= 2e^{r^2}\,\vec{r}\end{aligned}}$$

Q.48) $\phi = xy + yz + zx$

$$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$$

$$\nabla\phi = \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(y+x)$$

$\therefore$ point $A(1,2,0)$ पर —

$$\nabla\phi = 2\hat{i} + \hat{j} + 3\hat{k}$$

Now ~~$2\hat{i}+\hat{j}+3\hat{k}$~~ की दिशा $\hat{i} + 2\hat{j} + 2\hat{k}$ में इकाई सदिश

$$\vec{a} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{3}$$

Now फलन ϕ का $\vec{a}$ की दिशा में दिक् अवकलज

$$= \nabla\phi \cdot \vec{a}$$

$$\Rightarrow (2\hat{i} + \hat{j} + 3\hat{k})\left(\frac{\hat{i} + 2\hat{j} + 3\hat{k}}{9}\right) = \frac{10}{3}$$

~~page-8~~ यदि $\vec{a}$ व $\vec{b}$ दो अचर ~~सदिश~~ हो ~~,~~ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ हो तब–

(1) $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$

~~proof~~– $\nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

$$\therefore \boxed{\nabla(\vec{a}\cdot\vec{r}) = \vec{a}} \quad \text{या} \quad \boxed{\nabla(\vec{r}\cdot\vec{a}) = \vec{a}}$$

(2) $\nabla[\vec{r}\cdot\vec{a}\cdot\vec{b}] = \vec{a}\times\vec{b}$

$\Rightarrow \nabla\{\vec{r}\cdot(\vec{a}\times\vec{b})\} = \vec{a}\times\vec{b}$ {result ले}

(3) $\vec{a}\cdot\nabla = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

$$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$\therefore (\vec{a}\cdot\nabla)f = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)f$$

$$= a_1\frac{\partial f}{\partial x} + a_2\frac{\partial f}{\partial y} + a_3\frac{\partial f}{\partial z}$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{j})\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= \vec{a}\cdot\nabla f$$

(4) $(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$

$$\Rightarrow \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\Rightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\Rightarrow \vec{a}$$

~~page-11~~ (i) $\text{div}(\hat{r}) = \frac{2}{r}$

$$\Rightarrow \text{div}(\hat{r}) = \nabla\cdot\hat{r} = \nabla\cdot\frac{\vec{r}}{r}$$

$$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \sum \left\{ 1\cdot\frac{1}{r} + x\left(\frac{-1}{r^2}\right)\frac{x}{r} \right\}$$

$\frac{x}{r} = x\cdot\frac{1}{r}$

$$= \sum \frac{1}{r} - \frac{1}{r^3}\sum x^2$$

$$= \frac{3}{r} - \frac{1}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3}{r} - \frac{1}{r^3}(r^2) = \frac{2}{r}$$

अतः $\boxed{\text{div.}(\vec{r}) \text{ या } \nabla \vec{r} = \frac{2}{r}}$

Detail से-

$$\text{div.}(\hat{r}) = \nabla\cdot\hat{r} = \nabla\left(\frac{\vec{r}}{r}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \left\{1\cdot\frac{1}{r} + x\left(\frac{-1}{r^2}\right)\frac{x}{r}\right\} + \left\{1\cdot\frac{1}{r} + y\left(-\frac{1}{r^2}\right)\frac{y}{r}\right\} + \left\{1\cdot\frac{1}{r} + z\left(\frac{-1}{r^2}\right)\frac{z}{r}\right\}$$

$$= \frac{3}{r} - \frac{1}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

(iii) $\text{div}(r^n\vec{r}) = (n+3)r^n$

prove- $\text{div}(r^n\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{r^n(x\hat{i}+y\hat{j}+z\hat{k})\right\}$

$$= \sum \frac{\partial}{\partial x}(r^n\cdot x)$$

$$= \sum\left\{r^n\cdot 1 + x\cdot n\cdot r^{n-1}\cdot\frac{x}{r}\right\}$$

$$= \sum\left[r^n + n r^{n-2}x^2\right]$$

$$= \sum\left[r^n\right] + n r^{n-2}\sum x^2$$

$$= 3r^n + n r^{n-2}(r^2)$$

$$= 3r^n + n\cdot r^n$$

$$= (3+n)r^n$$

(vi) $\nabla \cdot \left(\frac{f(r)\, \vec{r}}{r} \right) = \frac{1}{r^2} \frac{d}{dr}(r^2 f)$

proof $L.H.S. = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left\{ \frac{f(r)}{r} (x\hat{i} + y\hat{j} + z\hat{k}) \right\}$

$= \sum \frac{\partial}{\partial x} \left\{ f(r) \cdot \frac{1}{r} \cdot x \right\}$

$= \sum \left\{ \frac{f(r)}{r} \cdot 1 + f(r)\, x \left(\frac{-1}{r^2} \right) \frac{x}{r} + \frac{f'(r)\, x}{r} \cdot \frac{x}{r} \right\}$

$= \sum \left[\frac{f(r)}{r} - \frac{f(r)\, x^2}{r^3} + \frac{f'(r)\, x^2}{r^2} \right]$

$= \sum \frac{f(r)}{r} - \frac{f(r)}{r^3} \sum x^2 + \frac{f'(r)}{r^2} \sum x^2$

$= \sum \frac{f(r)}{r} - \frac{f(r)}{r^3} \cdot r^2 + \frac{f'(r)}{r^2} \cdot r^2$

$= \sum \frac{f(r)}{r} - \frac{f(r)}{r} + f'(r)$

$= 3\frac{f(r)}{r} - \frac{f(r)}{r} + f'(r)$

$= 2\frac{f(r)}{r} + f'(r)$

$\therefore \boxed{\nabla \cdot \left(\frac{f(r)\, \vec{r}}{r} \right) = f'(r) + 2\frac{f(r)}{r}}$

$= \frac{1}{r^2}\left[r^2 f'(r) + 2r f(r) \right]$

$\boxed{\nabla \cdot \left(\frac{f(r)\, \vec{r}}{r} \right) = \frac{1}{r^2} \frac{d}{dr}(r^2 f)}$

(2) (iii) $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

prove $L.H.S = \nabla^2 f(r)$

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f(r)$

$= \sum \left(\frac{\partial^2}{\partial x^2} f(r) \right)$

$$= \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right)$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\cdot\frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left(f'(r)\cdot\frac{1}{r}\cdot x\right)$$

$$= \sum\left[\frac{f'(r)}{r}\cdot 1 + x f'(r)\left(\frac{-1}{r^2}\right)\frac{x}{r} + \frac{x}{r} f''(r)\frac{x}{r}\right]$$

$$= \sum \frac{f'(r)}{r} - \frac{f'(r)}{r^3}\sum x^2 + \frac{f''(r)}{r^2}\sum x^2$$

$$= \frac{3f'(r)}{r} - \frac{f(r)}{r^3}\cdot r^2 + \frac{f''(r)}{r^2}\cdot r^3$$

$$\boxed{\nabla^2 f(r) = \frac{2f'(r)}{r} + f''(r)}$$

Q.16) $\because$ $\vec{f}$ एक परिनालिका सदिश होगा यदि $div\vec{f} = 0$

$$\nabla\cdot\vec{f} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[(x+3y)\hat{i} + (x-2z)\hat{j} + (x+az)\hat{k}\right]$$

$$0 = 1 + 0 + a$$

$$\Rightarrow a = -1$$

Q.19) दिक् अवकलज पृष्ठ के लम्बवत महत्तम होता है।

Q.23) $\vec{f} = x^2z\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}$

$$\nabla\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2y^2z^2 & xy^2z \end{vmatrix}$$

$$= \hat{i}\left[-2y^2(2z) - 2xyz\right] - \hat{j}\left[-x^2 + y^2z\right] + \hat{k}\left[0-0\right]$$

$$(\nabla\times\vec{f})_{(1,1,-1)} = (-4y^2z - 2xyz)\hat{i} + (x^2 - y^2z)\hat{j}$$

$$= \left[-4(-1) + 2\right]\hat{i} + (1+1)\hat{j}$$

$$= 6\hat{i} + 2\hat{j}$$

Q.25) $\nabla\cdot(\nabla\times\vec{F}) \Rightarrow \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$

$= (2z-x)\hat{i} - \hat{j}(0-0) + \hat{k}(z-x^2)$

$\nabla\times\vec{F} = (2z-x)\hat{i} + (z-x^2)\hat{k}$

$\nabla\cdot(\nabla\times\vec{F}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[(2z-x)\hat{i} + (z-x^2)\hat{k}\right]$

$= \cancel{-\hat{i}+\hat{k}} \quad -1+1 = 0$

Q.27) $\int_0^1 (e^t\hat{i} + e^{-2t}\hat{j} + t\hat{k})\,dt$

$\Rightarrow \int_0^1 \cancel{\left[e^t - \frac{e^{-2t}}{2} + 1\right]_0^1}$

$\Rightarrow \cancel{\left[e^1 - \frac{e^{-2}}{2} + 1 - e^0 + \frac{e^0}{2} - 1\right]}$

$\Rightarrow \hat{i}\int_0^1 e^t\,dt + \hat{j}\int_0^1 e^{-2t}\,dt + \hat{k}\int_0^1 t\,dt$

$\Rightarrow \hat{i}\,[e^t]_0^1 + \hat{j}\left[\frac{e^{-2t}}{-2}\right]_0^1 + \hat{k}\left[\frac{t^2}{2}\right]_0^1$

$\Rightarrow \cancel{e^t\hat{i} - e^{-2t}}$

$(e^1 - e^0)\hat{i} + \left[\frac{e^{-2}}{-2} + \frac{e^{-0}}{+2}\right]\hat{j} + \left[\frac{1}{2} - 0\right]\hat{k}$

$\Rightarrow (e-1)\hat{i} + \frac{1}{2}(1-e^{-2})\hat{j} + \frac{1}{2}\hat{k}$

Q.30) $\because$ त्वरण $\vec{a} = \frac{dv}{dt} = t\hat{i} + 2\hat{j} + 3t^2\hat{k}$

Int. ~~diff.~~ w. r. to 't' $\rightarrow$

$v = \int(t\hat{i} + 2\hat{j} + 3t^2\hat{k})$

$v = \hat{i}\frac{t^2}{2} + 2t\hat{j} + \cancel{3}\frac{t^3}{\cancel{3}}\hat{k} + c \quad \text{—} (1)$

But जब $t = 0 \Rightarrow v = 2\hat{i} + \hat{j}$

$2\hat{i} + \hat{j} = c$

अतः by (1) $\Rightarrow$ $\boxed{v = \left(\frac{t^2}{2} + 2\right)\hat{i} + (2t+1)\hat{j} + t^3\hat{k}}$

Q.31) $\int_0^2 (\vec{r}\cdot\vec{s})\,dt$

$$\Rightarrow \int_0^2 [t\hat{i} - t^2\hat{j} + (t-1)\hat{k}]\cdot[2t^2\hat{i} + 6t\hat{k}]\,dt$$

$$\Rightarrow \int_0^2 (2t^3 + 6t^2 - 6t)\,dt$$

$$\Rightarrow \left[\frac{2t^4}{4} + \frac{6t^3}{3} - \frac{6t^2}{2}\right]_0^2$$

$$\Rightarrow \left[\frac{1}{2}(16) + 2(8) - 3(4)\right]$$

$$\Rightarrow [8 + 16 - 12] = 12$$

Q.32) $\int_0^2 (\vec{r}\times\vec{s})\,dt$

$$\vec{r}\times\vec{s} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -t^2 & (t-1) \\ 2t^2 & 0 & 6t \end{vmatrix}$$

$$= (-6t^3 - 0)\hat{i} - \hat{j}(6t^2 - 2t^3 + 2t^2) + \hat{k}(2t^4)$$

$$\int \vec{r}\times\vec{s} = \int_0^2 \left[(-6t^3)\hat{i} - (8t^2 - 2t^3)\hat{j} + (2t^4)\hat{k}\right]dt$$

$$= \hat{i}\left[-\frac{6t^4}{4}\right]_0^2 - \left(\frac{8t^3}{3} - \frac{2t^4}{4}\right)_0^2\hat{j} + \left(\frac{2t^5}{5}\right)_0^2\hat{k}$$

$$= -\frac{3}{2}(16)\hat{i} - \left[\frac{8}{3}(8) - \frac{1}{2}(16)\right]\hat{j} + \frac{2}{5}(32)\hat{k}$$

$$= -24\hat{i} - \left[\frac{64}{3} - 8\right]\hat{j} + \frac{64}{5}\hat{k}$$

$$= -24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}$$

Q.33) $\vec{F} = (x^2 - y^2)\hat{i} + xy\hat{j}$ $(0,0)$ to $(2,8)$

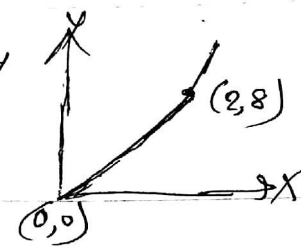
& curve $y = x^3$ —(1) $x = 0$ & $x = 2$

$y = 0$ & $y = 8$

$$\Rightarrow \int_C F\cdot dr = \int_C [(x^2 - y^2)\hat{i} + xy\hat{j}]\cdot[dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$= \int_C (x^2 - y^2)\,dx + \int_C xy\,dy$$

Now put the Value from ① $\Rightarrow$



$$\Rightarrow \int_0^2 (x^2 - x^6)\,dx + \int_0^8 y^{1/3} y\,dy$$

$$\Rightarrow \left[\frac{x^3}{3} - \frac{x^7}{7}\right]_0^2 + \left[\frac{3}{7} y^{7/3}\right]_0^8$$

$$\Rightarrow \left[\frac{8}{3} - \frac{8\times8\times2}{7}\right] + \frac{3}{7}(8)^{7/3}$$

$$\Rightarrow \left(\frac{8}{3} - \frac{128}{7}\right) + \frac{3}{7}\left[(2)^3\right]^{7/3}$$

$$\Rightarrow \left(\frac{56-384}{21}\right) + \frac{3}{7}(2^7)$$

$$\Rightarrow -\frac{328}{21} + \frac{128\times3}{7} = \frac{-328+1152}{21} = \frac{824}{21}$$

Q.51) $\int_C \vec{F}\cdot d\vec{r}$ given $y = x^3$ —①
$x = 0$ to $x = 2$
$y = 0$ to $y = 8$

$$\Rightarrow \int_C \left[(x^2 - y^2)\hat{i} + xy\hat{j}\right]\left[dx\hat{i} + dy\hat{j}\right]$$

$$\Rightarrow \int_C \left[(x^2 - y^2)\,dx + xy\,dy\right]$$

put Value from ① $\rightarrow$

$$\Rightarrow \cancel{(y^{2/3} - y^2)\,dx}$$

$$\Rightarrow \int_0^2 (x^2 - x^6)\,dx + \int_0^8 y^{1/3} y\,dy$$

$$\Rightarrow \left(\frac{x^3}{3} - \frac{x^7}{7}\right)_0^2 + \frac{3}{7} y^{7/3} + c$$

$$\Rightarrow \left(\frac{8}{3} - \frac{8\times8\times2}{7}\right) + \frac{3}{7}(2^3)^{7/3}$$

$$\Rightarrow \left(\frac{8}{3} - \frac{128}{7}\right) + \frac{3}{7}(8\times8\times2) = \frac{824}{21}$$

Q.43] $\int_C \vec{F}\cdot d\vec{r}$ given $y = x^2 - 4$
$x = 2$ to $x = 4$
$y = 0$ to $y = 12$

$$\Rightarrow \int_C xy\,dx + \int_C (x^2 + y^2)\,dy$$

$$\Rightarrow \int_2^4 x(x^2-4)\,dx + \int_0^{12} (y - 4 + y^2)\,dy$$