Toppersnotes
Unleash the topper in you

ENGINEERS ACADEMY

# RPSC - A.En.

## Assistant Engineering

## ELECTRICAL

Rajasthan Public Service Commission (RPSC)

## Volume - 10

## Electromagnetic Field Theory (EMFT)

SIERRA INNOVATIONS

# BASICS OF ELECTROMAGNETIC THEORY & MAXWELL'S EQUATIONS

## THEORY

## 1.1 Vector Algebra

There are 3-types of product

(i) Dot Product

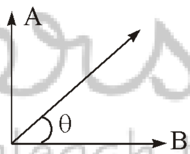
(ii) Cross Product

(iii) Triple Product

### *1.1.1 Vector Product*

**(i) Dot Product**

The Dot Product of two Vectors $\vec{A}$ and $\vec{B}$ is given by,




$$\vec{A}\cdot\vec{B} = A\cdot B\cdot\cos\theta$$

Let,
$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$
$$\vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$$

Thus, Dot product is given by

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Dot product is a Scalar quantity.

**(ii) Cross Product**

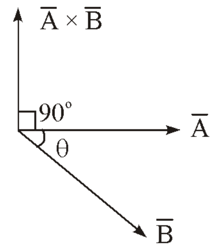
The Cross product of two Vectors $\vec{A}$ and $\vec{B}$ is given by

$$\vec{A}\times\vec{B} = |\vec{A}|\cdot|\vec{B}|\sin\theta\,\hat{a}_n$$

where, $\hat{a}_n$ = Normal unit vector. (Normal unit vector to AB plane)

$$\vec{A}\times\vec{B} = \hat{a}_x\left(A_yB_z - A_zB_y\right) - \hat{a}_y\left(A_xB_z - A_zB_x\right) + \hat{a}_z\left(A_xB_y - B_xA_y\right)$$

It is represented in the determinant form as given below




i.e.
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cross product is a Vector quantity.

**(iii) Triple Product**

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{C}.\vec{A}) - \vec{C}(\vec{A}.\vec{B})$$

### *1.1.2 Vector Operators*

(1) Gradient Operator is $\nabla V$ of scalar V

(2) Divergence operator is $\nabla.\vec{V}$ of vector $\vec{V}$

(3) Curl $\nabla \times A$ of vector $\vec{A}$

(4) Laplacian $\nabla^2 V$ of scalar V

**(1) Gradient ($\vec{\nabla}$ operator) :** Operator is given by $\nabla V$

Here del operator
$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

Gradient is applicable for Scalar fields only.

It given the Rate of change of Scalar field along the different co-ordinate axes.

***Example :*** Gradient of potential field V is given by

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

where, V is a Scalar field.

***Note :*** *Gradient of a potential field gives the electric field.*

i.e., $\vec{E} = -\vec{\nabla} \cdot V$, where, E is the electric field intensity.

***Note :*** *Gradient of a scalar field is a Vector quantity.*

**(2) Divergence :** It is applicable for a Vector field. Divergence of a Vector field gives the flux coming out of a closed surface, when volume of the surface shrinks to zero.

Let, $\vec{D} = D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z$ = Electric flux density

$$\vec{\nabla} \cdot \vec{D} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z\right)$$

$$\therefore \quad \vec{\nabla} \cdot \vec{D} = \frac{\partial}{\partial x}D_x + \frac{\partial}{\partial y}D_y + \frac{\partial}{\partial z}D_z$$

The above equation represent the divergence of a Vector quantity $(\vec{D})$.

***Note :*** *Divergence of a Vector field is a Scalar field.*

***Example :*** $\vec{\nabla} \cdot \vec{D} = \rho_v$ = Charge density

**(3) Curl of a Vector field :** The curl of a Vector field.

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z \text{ is given by}$$

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\hat{a}_x - \left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right)\hat{a}_y + \left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)\hat{a}_z$$

For example, The curl of Magnetic field intensity $(\vec{H})$ represented as

$$\vec{H} = H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z$$

Can be given by determinant form as shown below

$$\vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y\right)\hat{a}_x - \left(\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial z}H_x\right)\hat{a}_y + \left(\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right)\hat{a}_z$$

***Note :*** *Curl of a Vector field is a Vector quantity.*

***Example :*** $\vec{\nabla} \times \vec{H} = J$ = Current density.

**(4) Laplacian ($\nabla^2$) :** Laplacian of scalar V is divergence of gradient of scalar V

$$\nabla^2 V = \nabla.\nabla V = \text{Divergence (Gradient V)}$$

For Cartesian Coordinate

Laplacian $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

***Note :*** *A vector $\vec{A}$ is said to be solenoidal if its divergence is zero*

$$\nabla.\vec{A} = 0$$

***Example :*** Magnetic field is solenoidal

$$\nabla.\vec{B} = 0$$

*A vector $\vec{A}$ is said to be irrotational if its curl is zero.*

$$\nabla \times \vec{A} = 0$$

***Example :*** In static environment Electric field is irrotational or conservative

$$\nabla \times E = 0$$

- *A scalar field is said to be harmonic in given region if its laplacian is zero.*

$$\nabla^2 V = 0$$

- *Divergence of curl is always zero*

$$\nabla.(\nabla \times A) = 0$$

- *Curl of gradient is always zero*

$$\nabla \times (\nabla.A) = 0$$

### *1.1.3 Divergence Theorem*

According the divergence theorem, "The surface integral of a vector field over a closed surface S is equal to the Volume integral of divergence of the Vector field".

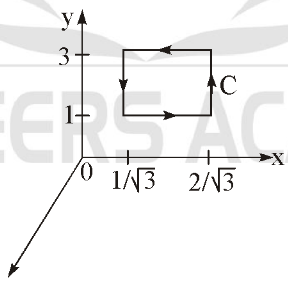
$$\oint_s \vec{D} \cdot \vec{ds} = \int_v \vec{\nabla} \cdot \vec{D}\, dv$$

### *1.1.4 Stokes Theorem*

According to this theorem, "The Line integral of a Vector field over a closed path is equal to the Surface integral of curl of the Vector field".

$$\oint_\ell \vec{H} \cdot \vec{d\ell} = \int_s (\vec{\nabla} \times \vec{H}) \cdot \vec{ds}$$

***Example 1 :*** Given vector field $\vec{A} = xy\hat{a}_x + x^2\hat{a}_y$. Find $\oint_c A\,dl$ circulation by stoke's theorem over path given below.






***Solution :***

Stoke's theorem $\qquad \oint_c A\,dl = \iint (\nabla \times A)\,ds$

Curl
$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & 0 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x^2)\right]a_x + \left[\frac{\partial}{\partial z}(xy) - 0\right]a_y + \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right]a_z$$

$$\nabla \times A = x.\hat{a}_z$$

area element $ds = dx \,.\, dy \,.\, \hat{a}_z$

Using stokes' theorem

$$\oint A.dl = \iint(\nabla \times A).ds$$

$$= \int_1^3 \int_{1/\sqrt{3}}^{2/\sqrt{3}} x.\,dx\,dy = 1$$

***Example 2 :*** Given the vector field $A = y^2a_x + (2xy + x^2 + z^2)\,a_y + (4x + 2yz)\,a_z$. Find divergence of vector field.

***Solution :***

Divergence is given by $\nabla.\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$

$$= \frac{\partial}{\partial x}[y^2] + \frac{\partial}{\partial y}[2xy + x^2 + z^2] + \frac{\partial}{\partial z}[4x + 2yz]$$

$$= 0 + 2x + 2y$$

$$= 2(x + y)$$

***Example 3 :*** A scalar field $g = (1 + 2k)x^2y + xyz$ will be harmonic at all point for which value of k.

***Solution :***

Condition for harmonic field $\nabla^2 g = 0$

$$\nabla^2 g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0$$

$$= \frac{\partial}{\partial x}[2x(1+2k)y + yz] + \frac{\partial}{\partial y}[(1+2k)x^2 + xz] + \frac{\partial}{\partial z}[xy]$$

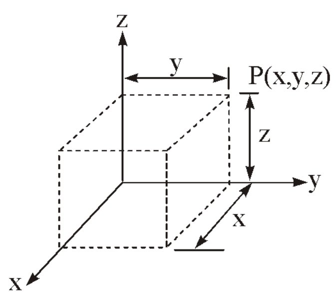
$$= 2(1+2k) + 0 + 0 = 0$$

Therefore $k = -\frac{1}{2}$

## 1.2 Co-ordinate Systems

### *1.2.1 Cartesian Co-ordinate System*

The co-ordinates of a point P in the cartesian co-ordinate system is x, y and z along the x, y and z axis. It can be represented as P (x, y, z) as shown below



- Differential length in cartesian coordinate is

$$\vec{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

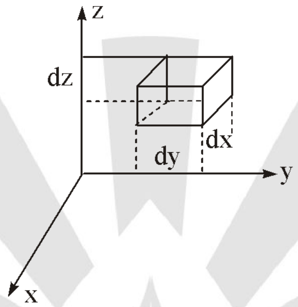
- Differential area in cartesian co-ordinates

$$\vec{ds_1} = dy\,dz\cdot\hat{a}_x$$

$$\vec{ds_2} = dx\,dz\cdot\hat{a}_y$$

$$\vec{ds_3} = dx\,dy\cdot\hat{a}_z$$

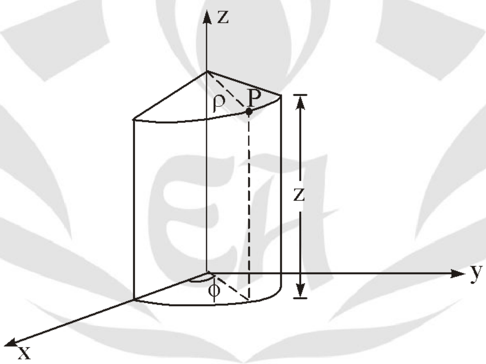
- Differential volume in cartesian co-ordinates is




$$dv = dxdydz$$

### *1.2.2 Cylindrical Co-ordinate System*

The cylindrical co-ordinates of a point is represented in terms of ρ, ϕ and z along the cylinder as given below




Here, ρ = Radius of cylinder.
ϕ = Angle between x-axis and perpendicular on x-axis of the point.

- Differential volume in cylindrical co-ordinates is given by

$$dv = \rho d\rho.d\phi.dz$$

- Differential Length in cylindrical co-ordinates is given by

$$d\ell = d\rho\hat{a}_\rho + \rho d\phi\hat{a}_\phi + dz\hat{a}_z$$

- Differential Area in cylindrical co-ordinates is given by

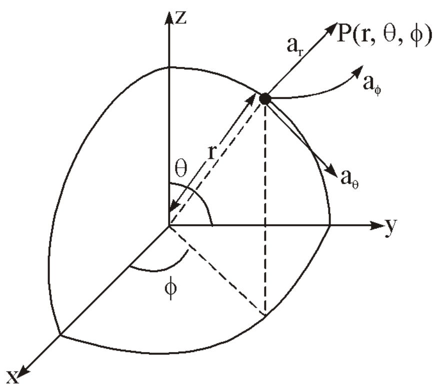
$$\vec{ds_\rho} = \rho d\phi dz\cdot\hat{a}_\rho$$

$$\vec{ds_\phi} = d\rho\cdot dz\cdot\hat{a}_\phi$$

$$\vec{ds_z} = (d\rho)(\rho d\phi)\cdot\hat{a}_z$$

### *1.2.3 Spherical Co-ordinate System*

The spherical co-ordinates of a point is represented interms of r, θ & φ along the spherical surface as shown below :




Differential length

$$dl = dr\,\hat{a}_r + rd\theta\,\hat{a}_\theta + r\sin\theta.d\phi\,\hat{a}_\phi$$

Differential area

$$ds = \begin{cases} r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r \\ r\sin\theta\,dr\,d\phi\,\hat{a}_\theta \\ r\,dr\,d\theta\,\hat{a}_\phi \end{cases}$$

Differential volume $dv = (dr)\ (rd\theta)\ (r\ \sin\ \theta\ d\phi)$

### *1.2.4 General Co-ordinate System : (U, V, W)*

| | U | V | W | $h_1$ | $h_2$ | $h_3$ |
|---|---|---|---|---|---|---|
| Rectangular | x | y | z | 1 | 1 | 1 |
| Cylindrical | ρ | φ | z | 1 | ρ | 1 |
| Spherical | r | θ | φ | 1 | r | r sin θ |



**Mathematical Expressions of Operators**

(i) Gradient
$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial V}{\partial v} + \frac{1}{h_3}\frac{\partial V}{\partial w}$$

(ii) Divergence of Vector
$$\vec{A} = A_u\hat{a}_u + A_v\hat{a}_v + A_w\hat{a}_w$$

$$\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}[h_2h_3A_u] + \frac{\partial}{\partial v}(h_3h_1A_v) + \frac{\partial}{\partial w}(h_1h_2A_w)\right]$$

(iii) Laplacian
$$\nabla^2 V = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial V}{\partial w}\right)\right]$$

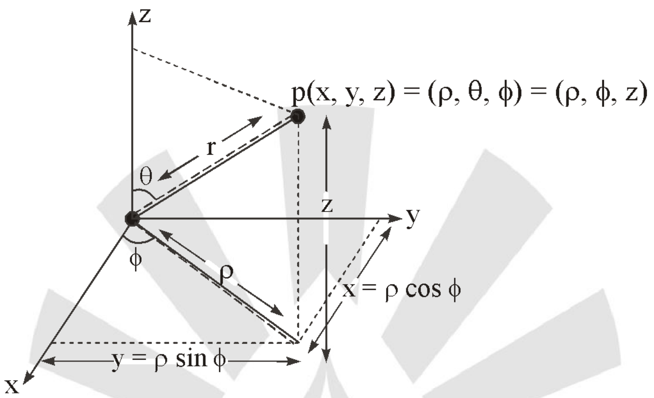
(iv) Curl
$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_u & h_2\hat{a}_v & h_3\hat{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1A_u & h_2A_v & h_3A_w \end{vmatrix}$$

(v) Area $$ds = \begin{cases} h_2h_3 & \partial v\partial w & \hat{a}_u \\ h_1h_3 & \partial u\partial w & \hat{a}_v \\ h_1h_2 & \partial v\partial w & \hat{a}_w \end{cases}$$

(vi) Volume $$dv = h_1 h_2 h_3 \partial u\ \partial v\ \partial w$$

(vii) Length $$dl = h_1 du\ \hat{a}_u + h_2 dv\ \hat{a}_v + h_3 dw\ \hat{a}_w$$

### *1.2.5 Co-ordinate Transformation*




Relation between Cylindrical and cartesian co-ordinates

| ***Cylindrical*** | ***Cartesian*** |
|---|---|
| $\rho = \sqrt{x^2 + y^2}$ | $x = \rho \cos \phi$ |
| $\phi = \tan^{-1}\left[\frac{y}{x}\right]$ | $y = \rho \sin \phi$ |
| $z = z$ | |

Relation between spherical and other co-ordinates

| ***Spherical*** | ***Other Co-ordinates*** |
|---|---|
| $r = \sqrt{x^2 + y^2 + z^2}$ | $\rho = r \sin \theta$ |
| $\theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right]$ | $x = r \sin \theta \cos \phi$ |
| $\phi = \tan^{-1}\left(\frac{y}{x}\right)$ | $y = r \sin \theta \sin \phi$ |

***Example 4 :*** Determine divergence of vector fields

(a) $\vec{A} = \rho \sin \phi\ \hat{a}_\rho + \rho^2 z\ \hat{a}_\phi + z\cos\phi\ \hat{a}_z$

(b) $\vec{B} = \frac{1}{r^2}\cos \theta\ \hat{a}_r + r\sin^2\theta\cos\phi\ \hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

***Solution :***

(a) $$\nabla.\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(A_\phi) + \frac{\partial}{\partial z}A_z$$

$$= \frac{1}{\rho}\frac{1}{\partial\rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}[z\cos\phi]$$

$$= 2\sin\phi + \cos\phi$$

(b)
$$\nabla.\vec{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(B_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(B_\phi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta)$$

$$= 0 + 2\cos\theta\cos\phi + 0$$

$$= 2\cos\theta\cos\phi$$

***Example 5 :*** For above vector field $\vec{A}$ find curl $\nabla\times\vec{A}$

***Solution :***

$$\nabla\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$= \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho\sin\phi & \rho^2 z & z\cos\phi \end{vmatrix}$$

$$= \left[\frac{z}{\rho}\sin\phi - \rho^2\right]\hat{a}_\rho + 0 + \frac{1}{\rho}[3\rho^2 z - \rho\cos\phi]\hat{a}_z$$

$$= -\frac{1}{\rho}(z\sin\phi + \rho^3).\hat{a}_\rho + (3\rho z - \cos\phi)a_z$$

## 1.3 Electrostatics



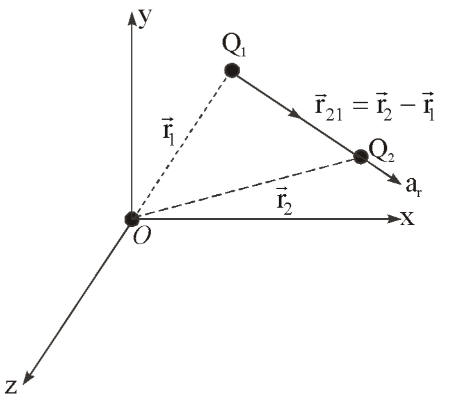
Stationary charge produces electric field $\vec{E}$.

A charge may be point charge, line charge, surface charge or volume charge distributed.

There are two laws in electrostatics **coulomb's law and gauss law.**

### *1.3.1 Coulomb's Law*

***Statement :*** The force between two point charge $Q_1$ and $Q_2$ is inversely proportional to square of distance between two charges and directed along the vector connecting two charges.



Force, $$F = \frac{kQ_1Q_2}{|\vec{r}_{21}|^2}.\hat{a}_r$$

$$F_{21} = \frac{Q_1Q_2.(\vec{r}_2 - \vec{r}_1)}{4\pi \in_0 |\vec{r}_2 - \vec{r}_1|^3}$$

Electric field $\vec{E}$ intensity is defined as force per unit charge

$$\vec{E} = \frac{\vec{F}}{Q}$$

- Electric field due to point charge

$$\vec{E} = \frac{Q}{4\pi \in_0 r^2}.\hat{a}_r$$

- Electric field due to line charge

$$\vec{E} = \frac{\int \rho_L dl}{4\pi \in_0 r^2}\hat{a}_r$$

- Electric field due to surface charge

$$\vec{E} = \frac{\iint \rho_s ds}{4\pi \in_0 r^2}.\hat{a}_r$$

- Electric field due to volume charge

$$\vec{E} = \frac{\iiint \rho_v dv}{4\pi \in_0 r^2}.\hat{a}_r$$

Electrostatic potential is defined as work done per unit charge and it is scalar potential due to point charge.

$$V = \frac{Q}{4\pi \in_0 r}$$

Gradient of potential is electric field.

$$\vec{E} = -\nabla V$$

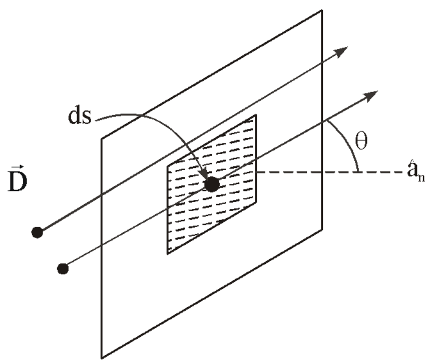
For close loop 'C' work done is zero

$$V = -\oint_C E.dl = 0$$

by stokes theorem for static field.

$$\nabla \times \vec{E} = 0$$

Electric flux passing through any surface areas



Electric flux $\Psi = \iint D.ds$

where, D = Electric field density c/m$^2$.

### *1.3.2 Gauss's Law*

***Statement :*** The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

i.e., $\Psi = \oint_S D.ds = Q_{enclosed}$

Integral form $\oint_S D.ds = \int_v \rho_v dv$ $(\rho \neq \rho_v)$

$\rho_v$ = Volume charge density

Differential form $\nabla.D = \rho_v$

***Example 6 :*** Charge density inside a hollow spherical shell of radius r = 4 cm centered at origin defined as

$$\rho_V = \begin{cases} 0 & \text{for} \quad r \leq 2 \\ \dfrac{4}{r^2} c/m^3 & \text{for} \quad 2 < r \leq 4 \end{cases}$$

Find Electric field intensity at r = 3

***Solution :***

From Gauss law $\oint E.ds = \dfrac{Q_{enc}}{\epsilon_0} = \dfrac{1}{\epsilon_0}\int \rho_v dv$

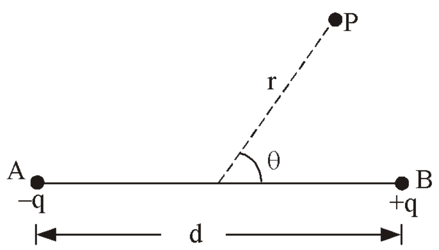
$$= \frac{1}{\epsilon_0}\int (0)dv + \frac{1}{\epsilon_0}\int \frac{4}{r^2}dv \quad [0 < r \leq 3]$$

$$E(4\pi R^2) = \frac{1}{\epsilon_0}\int_{r=2}^{3}\int_0^{\pi}\int_0^{2\pi} \frac{4}{r^2}\,[r^2 \sin\theta \; dr \; d\theta \; d\phi]$$

$$E\,(4\pi \times 3^2) = \frac{4\pi \times 4}{\epsilon_0}(3-2)$$

$$E = \frac{4}{9\epsilon_0} a_r$$

### *1.3.3 Electric Dipole*



- Electric dipole consist of two point charge, separated by small distance d having opposite polarity.
- Dipole moment $p = qd$
- Electric potential due to dipole is given by

$$V = \frac{P\cos\theta}{4\pi \epsilon_0 r^2}$$

***Note :*** *Potential is maximum along dipole and it is inversely proportional to square of distance.*

Electric field due to dipole is given by

$$\vec{E} = \frac{P}{4\pi \epsilon_0 r^3}[2\cos\theta\, a_r + \sin\theta\, a_\theta]$$

***Note :*** *For monopole* $\vec{E} \propto \frac{1}{r^2}$

*For Dipole* $\vec{E} \propto \frac{1}{r^3}$

### 1.3.4 Electrostatic Energy

Energy stored in the system with electric field E and electric flux density $\vec{D}$ is given by

$$W_e = \frac{1}{2}\int_v \vec{D}.\vec{E}.dv$$

$$= \frac{1}{2}\int_v \epsilon_0 E^2 dv$$

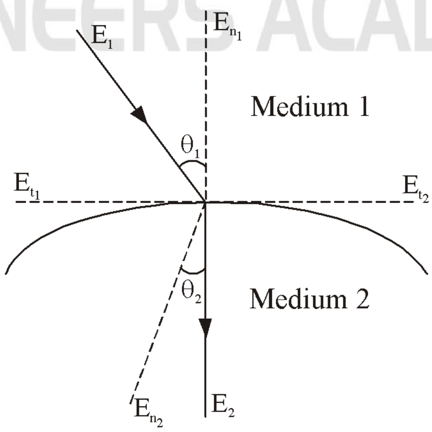
### 1.3.5 Electric Boundary Conditions

Boundary conditions are defined when region consist of two different media.

Electric field composed of two orthogonal component, tangential component $E_t$ and normal component $E_n$.

$$E = E_t + E_n$$

Consider the two different dielectric medium (1) and (2) with permittivities $\epsilon_1$ and $\epsilon_2$ respectively as shown below




According to boundary condition, tangential component of electric field is continuous at boundary,

i.e.,
$$E_{t_1} = E_{t_2} \text{ or } \frac{D_{t_1}}{\epsilon_1} = \frac{D_{t_2}}{\epsilon_2}$$

If the surface charge density at boundary is $e_s$ then boundary condition becomes.

$$D_{n_1} - D_{n_2} = e_s$$

### *1.3.6 Poission's and Laplace Equation*

Electric potential V and volume charge density $E_v$ in certain region is related by poissions equation

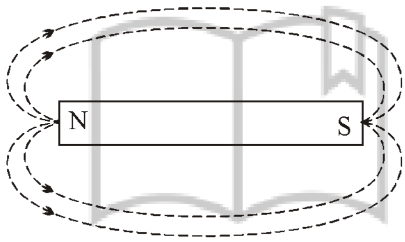
$$\nabla^2 V = \frac{\rho_V}{\epsilon}$$

For charge free region
$$\nabla^2 V = 0$$

Uniqueness theorem states that if solution of Laplace or poission equation satisfies the boundary condition, then solution is 'Unique'.

## 1.4 Magnetostatic Fields

Magnetic field is produced by moving charges or constant current flow.




Magnetic flux is concentration of magnetic flux line outward from north pole towards south pole of magnet.

Magnetic flux density is defined as magnetic flux per unit area and it is vector quantity. Its unit is Tesla (T) or weber per squared meter (1 wb/m$^2$)

$$\vec{B} = \frac{d\Psi}{ds} a_n$$

Flux
$$\Psi = \int_s B.ds$$

Relation between magnetic flux density $\vec{B}$ and magnetic field intensity $\vec{H}$ is given by

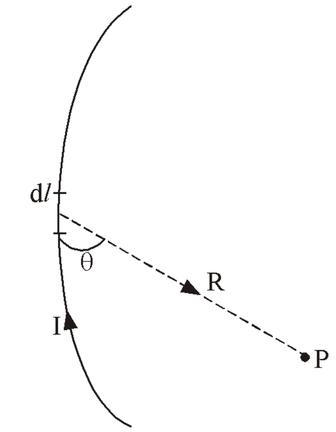
$$B = \mu H = \mu_0 \mu_r H$$

where, $\mu$ = Permeability of medium

$\mu_0 = 4\pi \times 10^{-7} H/m$

### *1.4.1 Bio-Savart's Law*

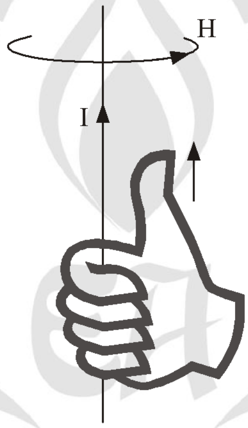
***Statement :*** The magnetic field intensity dH produces at point P due to current element Id*l* is given by




$$dH = \frac{Idl \sin\theta}{4\pi R^2}$$

or
$$dH = \frac{Idl \times a_R}{4\pi R^2} = \frac{Idl \times \vec{R}}{4\pi R^3}$$

Direction of magnetic field is given by right-hand rule where fingers shows magnetic field line and direction of thumb show current I.




### 1.4.2 Ampere's Law

***Statements :*** Line integral of magnetic field intensity around any closed path is equal to current enclosed by the path

$$\oint_l H.dl = I_{enclosed}$$

By stoke's theorem

$$\oint_l H dl = \iint_s (\nabla \times H).ds = \iint J.ds$$

or
$$\nabla \times H = J$$

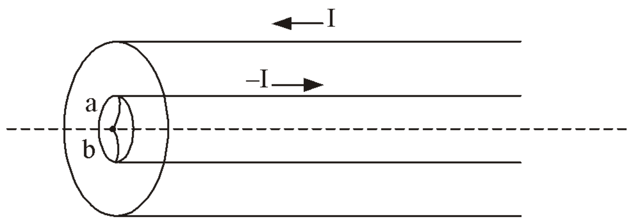
Curl of magnetic field intensity $\vec{H}$ is equal to current density J.

***Example 7 :*** Consider hollow concentric cylinder carrying I and –I current in opposite direction. Find magnetic field intensity inside and outside cylinder.

***Solution :***

**Case-I :** If r < a

using ampere's Law




$$\oint H dl = I_{enc.}$$

Inside inner cylinder current enclosed is zero

$$\oint H dl = 0$$

H = 0 inside inner cylinder.

**Case-II :** If a < r < b

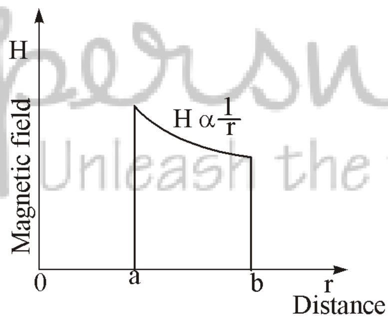
$$\oint H dl = I$$

$$H = \frac{I}{2\pi r^2} a_\phi$$

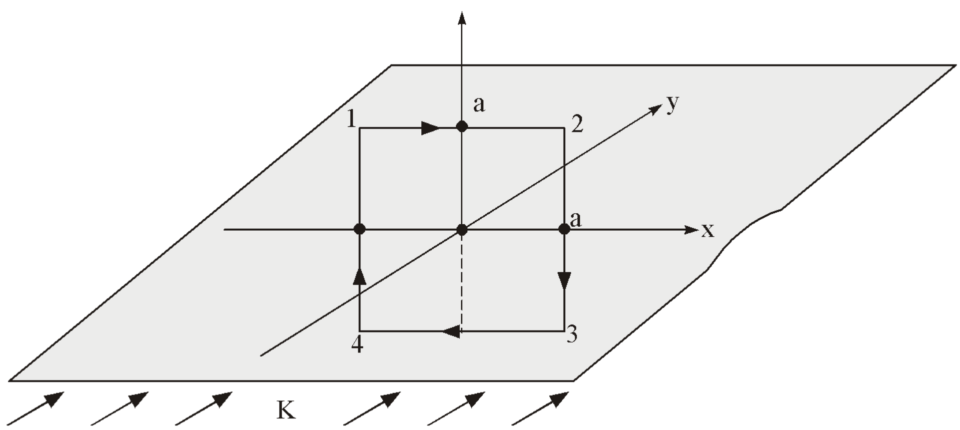
**Case-III :** If r > b

$$\oint H dl = I - I = 0$$

H = 0 outside outer cylinder




***Example 8 :*** An infinite current sheet lies in the z = 0 plane with $K = ka_y$, as shown in figure. Find H.



***Solution :***

The Biot-Savart law and considerations of symmetry shown that H has only an x component, and is not a function of x or y.

Apply Ampere's Law to the square contour 2341, and using the fact that H must be antisymmetric in z,

$$\oint H.dl = (H)(2a) + 0 + (H)(2a) + 0$$

$$= (K)(2a) \quad \text{or} \quad H = \frac{K}{2}$$

Thus for all z > 0, $$H = \left(\frac{K}{2}\right)a_x.$$

More generally, for an arbitrary orientation of the current sheet,

$$H = \frac{1}{2}K \times a_n$$

$a_n$ is the unit vector perpendicular to the plane of the sheet.

Observe that H is independent of the distance from the sheet. Further, the directions of H above and below the sheet can be found by applying the right-hand rule to a few of the current elements in the sheet.

### *1.4.3 Magnetic Potential*

There are two type of magnetic potentials

**(1) Magnetic scalar potential ($V_m$)**

$$H = -\nabla V_m$$

or $$(V_m) = \int_y^x H.dl$$

For source free region (J = 0), then magnetic scalar potential satisfies the Laplace's equation

i.e., $$\nabla^2 V_m = 0$$

It is only defined for current free region.

**(2) Magnetic Vector Potential $(\vec{A})$ :**

Magnetic field density $\vec{B}$ can be expressed as curl of magnetic vector potential $\vec{A}$

$$\vec{B} = \nabla \times \vec{A}$$

Magnetic vector potential satisfies the poission's equation

i.e., $$\nabla^2 A = -\mu_0 J$$

***Example 9 :*** Find current density that would produce magnetic vector potential $A = 2a_\phi$ in cylindrical co-ordinate.

***Solution :***

Magnetic flux density is given by

$$B = \nabla \times A$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)a_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho)a_z$$

$$B = \frac{2}{\rho}a_z$$

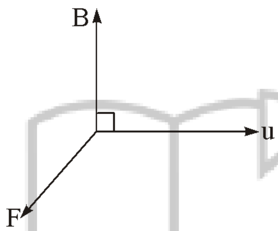
$$J = \frac{1}{\mu_0}(\nabla \times B)$$

$$= \frac{1}{\mu_0}\left[-\frac{\partial}{\partial \rho}\left(\frac{2}{\rho}\right)\right]a_\phi$$

$$= \frac{2}{\mu_0 \rho^2}a_\phi$$

### *1.4.4 Force in Magnetic Field*

If a charge particle 'Q' is in motion with velocity 'v' in presence of magnetic field density 'B' the magnetic force experience by charge is




$$F = Q(v \times B)$$

If both magnetic field and electric field are present force on charged partical is given by 'Lorentz force.'

$$F = Q(E + v \times B)$$

Magnetic force can not perform work while electric force can perform work.

### *1.4.5 Magnetic Dipole*

Magnetic dipole moment is product of current and area A, and it is normal to plane of Loop.

$$m = I.A\ a_n$$

### *1.4.6 Magnetization in Magnetic Material*

Magnetization is directly proportional to magnetic field intensity.

i.e., $M \alpha H$

$$M = X_m H$$

where, $X_m$ is magnetic susceptibility and it is given by

$$X_m = \mu_r - 1$$

where, $\mu_r$ = Relative permeability of medium.

### *1.4.7 Magnetic Energy*

In a magnetic field density B, stored magnetic energy density is given by

$$W_m = \frac{1}{2}(B.H) = \frac{1}{2}\mu H^2 \qquad \left[\because \vec{B} = \mu\vec{H}\right]$$

### *1.4.8 Magnetic Boundary Condition*

In two different magnetic media with permeabilities $\mu_1$ and $\mu_2$ respectively, at boundary field components are given by boundary condition.

From the boundary condition, the normal components of magnetic field are related.

$$B_{1_n} = B_{2_n}$$

or normal component of magnetic field density are equal at boundary.

Tangential components of magnetic field intensity are related as

$$H_{1_t} - H_{2_t} = K \text{ or } (H_1 - H_2) \times a_{n12} = K$$

where, K is surface charge density.

If K = 0

$$H_{1_t} = H_{2_t}$$

## 1.5 Maxwell's Equation

### *1.51 Faraday Law*

**Faraday's Law of Electro-magnetic Induction :** The –ve sign indicates that the polarity of voltage induced opposes the cause of induction of the voltage.

This is as per the Lenz's Law.

$$e = \oint E \cdot d\ell.$$

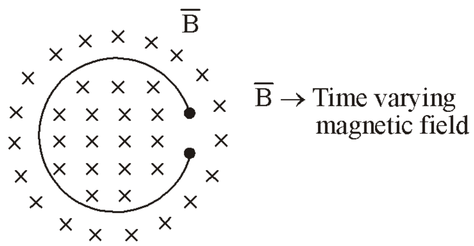
also,
$$e = \frac{d\phi}{dt}$$

Since the voltage induced in the loop is in the closed path; therefore it can be related to the electric field as under

$$e = \oint \overline{E} \cdot \overline{d\ell}$$

"Electric potential induced in a loop, is equal to the Rate of change of magnetic flux linking with the loop".

$$e = -\frac{d\phi}{dt}$$




The Magnetic flux ($\phi$) in terms of Magnetic flux density $(\vec{B})$ can be given by

$$\phi = \int_s \vec{B} \cdot \vec{dS} \qquad \text{...(i)}$$

Putting these expressions in equation (i), we have

$$\oint_{\ell} \vec{E} \cdot \overrightarrow{d\ell} = -\frac{d}{dt}\left[\int_{s} \vec{B} \cdot \overrightarrow{dS}\right]$$

$$\oint_{\ell} \vec{E} \cdot \overrightarrow{d\ell} = \int_{s}\left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot \overrightarrow{dS}$$

**The above equation gives**

Integral form of faraday's Law

Maxwell's equation in integral form Now, as per Stoke's theorem

$$\oint_{\ell} \vec{E} \cdot \overrightarrow{d\ell} = \int (\vec{\nabla} \times \vec{E}) \cdot \overrightarrow{dS}$$

or, $$\int_{s} (\vec{\nabla} \times \vec{E}) \cdot \overrightarrow{dS} = \int_{s}\left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot \overrightarrow{dS}$$

∴ $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

**The above equation gives**

Point form or differential form of Faradays' Law

Maxwell's equation.

### *1.5.2 Maxwell's Equations*

***Maxwell's Equations for Time varying fields***

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

But $$\vec{B} = \mu \vec{H}$$

where, $\mu$ = permeability of the medium.

Permeability ($\mu$) is given by

$$\mu = \mu_0 \mu_r = 4\pi \times 10^{-7} \text{ H/m} \quad \text{(for free space)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\mu \vec{H})$$

$$\vec{\nabla} \times \vec{E} = \mu \frac{\partial \vec{H}}{\partial t}$$

- $$\vec{\nabla} \cdot \vec{E} = \rho_v$$

But, $$\vec{D} = \varepsilon \vec{E}$$

where, $\varepsilon$ = permittivity of the medium

Permittivity of the medium (ε) is given by

$$\varepsilon = \varepsilon_0\varepsilon_r;\ \varepsilon_0 = 8.85\times10^{-12}\,\text{F/m} \qquad \text{(for free space)}$$

So, $$\vec{\nabla}\cdot\vec{E} = \frac{\rho_v}{\varepsilon}$$

• $$\vec{\nabla}\cdot\vec{B} = 0 \text{ But, } \vec{B} = \mu\vec{H}$$

∴ $$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

• $$\vec{\nabla}\times\vec{H} = \vec{J}_C + \vec{J}_D$$

where, $\vec{J}_D$ is displacement current

and $\vec{J}_D = \frac{\partial\vec{D}}{\partial t}$ ($\vec{D}$ is the displacement current density)

and $\vec{D} = \varepsilon\vec{E}$

Therefore, $\vec{J}_D$ is given by,

$$\vec{J}_D = \frac{\partial\vec{D}}{\partial t} = \varepsilon\frac{\partial\vec{E}}{\partial t}$$

Thus, $$\vec{\nabla}\times\vec{H} = \vec{J}_C + \vec{J}_D = \vec{J}_C + \frac{\partial\vec{D}}{\partial T}$$

**Maxwell's Equations for free space**

For free space, $\varepsilon = \varepsilon_0;\ \mu = \mu_o$

and Conduction current $\vec{J}_C = 0$ (Because there are no charge carriers in free space)

∴ $\rho_v = 0$

Thus, the Maxwell's equation are

$$\vec{\nabla}\cdot\vec{D} = 0;\ \vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0;\ \vec{\nabla}\cdot\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_o\frac{\partial\vec{H}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_C + \frac{\partial\vec{D}}{\partial t} = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

**Maxwell's equations for perfect dielectric**

For perfect dielectrics Charge density

$$\rho_v = 0 \text{ and } \vec{J}_C = 0$$

(There are no free charge carriers is perfect dielectric)

Thus, the Maxwell's equation are

$$\vec{\nabla}\cdot\vec{D} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = \frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_D = \frac{\partial\vec{D}}{\partial t} = \varepsilon\frac{\partial\vec{E}}{\partial t}$$

**Maxwell's Equations for Harmonically Time Varying Fields**

- For a Harmonically Time-Varying field

$$E = E_0(\cos\omega t + j\sin\omega t)$$

$$= E_0 e^{j\omega t}$$

$$E = E_0 e^{j\omega t} \text{ and } H = H_0 e^{j\omega t}$$

$$\frac{\partial\vec{E}}{\partial t} = E_0 \times j\omega e^{j\omega t} \text{ and } \frac{\partial H}{\partial t} = j\omega H_0 e^{j\omega t}$$

$$\frac{\partial\vec{E}}{\partial t} = j\omega\vec{E} \text{ and } \frac{\partial\vec{H}}{\partial t} = j\omega\vec{H}$$

From the above equations, For Harmonically Time-Varying fields,

$$\frac{\partial}{\partial t} = j\omega$$

So, the Maxwell's equations are

$$\vec{\nabla}\cdot\vec{D} = \rho, \text{ or } \vec{\nabla}\cdot\vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -j\omega\vec{B}$$

or $$\vec{\nabla}\times\vec{E} = -j\omega\mu\vec{H}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_C = \frac{\partial\vec{D}}{\partial t} = j\omega\vec{D} = j\omega\vec{D} = \vec{J}_C + j\omega\varepsilon\vec{E}$$

But, the conduction current density $(\vec{J}_C)$ is

$$\vec{J}_C = \sigma\vec{E} \text{...(Ohm's Law)}$$

$$\therefore \quad \vec{\nabla}\times\vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E} \text{ or, } \vec{\nabla}\times\vec{H} = (\sigma + j\omega\varepsilon)\vec{E}$$

**Maxwell's equation for Harmonically Time-Varying field in a perfect dielectric**

For a perfect dielectric,

$$\rho_v = 0 \text{ and } \vec{J}_C = 0$$

- Thus, the Maxwell's equation are

$$\vec{\nabla}\cdot\vec{D} = 0$$

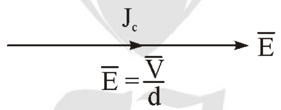
$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -j\omega\mu\vec{H} = -j\omega\vec{B}$$

$$\vec{\nabla}\times\vec{H} = j\omega\vec{D} = j\omega\varepsilon\vec{E}$$

### *1.5.3 Conduction and Displacement Currents Densities*

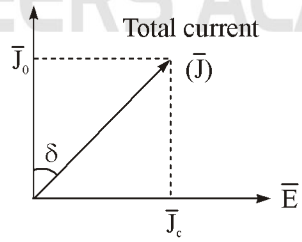
- According to Ohm's Law,
- Conduction Current density $\left(\vec{J}_C\right)$ is given by $\vec{J}_C = \sigma\vec{E}$
- Displacement current density $\left(\vec{J}_D\right)$ is given by $\vec{J}_D = \frac{\partial\vec{D}}{\partial t} = \varepsilon\frac{\partial\vec{E}}{\partial t}$
- For harmonically time-varying fields :




$$\vec{J}_D = j\omega\varepsilon\vec{E}$$

**Loss tangent (tan δ)**

$$\tan\delta = \left|\frac{\vec{J}_C}{\vec{J}_D}\right|$$




$$\text{Loss tangent} = \frac{|\sigma\vec{E}|}{|j\omega\varepsilon\vec{E}|} = \frac{|\sigma|}{|j\omega\varepsilon|}$$

$$\text{Loss tangent} = \tan\delta = \frac{|\sigma|}{|j\omega\varepsilon|}$$

$$\therefore \qquad \frac{\vec{J}_C}{\vec{J}_D} = \frac{\sigma}{j\omega\varepsilon},$$

$$\vec{J}_C = \frac{\sigma}{j\omega\varepsilon}\cdot\vec{J}_D$$

$$\text{or} \qquad \vec{J}_C = \frac{\sigma}{\omega\varepsilon}\cdot\vec{J}_D \angle -90^\circ$$

Conduction current density $(\vec{J}_C)$ lags the displacement current density $(\vec{J}_D)$ by $90^\circ$.

**Classification of Dielectrics on the basics of the dielectric behaviors**

**Case I :** Good dielectrics. For a good dielectric,

$$\vec{J}_C << \vec{J}_D$$
$$\sigma << \omega\varepsilon$$

**Case II :** Semi-dielectrics. For a semi-dielectric,

$$\vec{J}_C = \vec{J}_D \text{ and } \sigma = \omega\varepsilon$$

$$\text{loss tangent} = \left|\frac{\vec{J}_C}{\vec{J}_D}\right| = 1$$

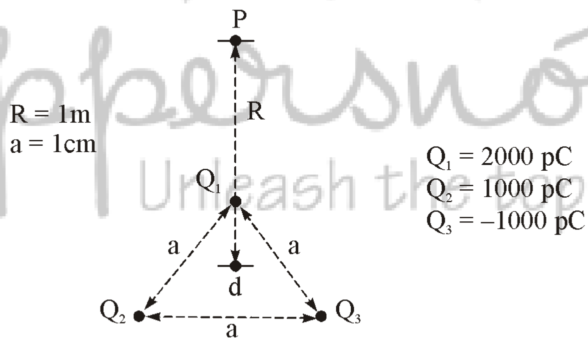
$$\Rightarrow \qquad \tan\delta = \tan 45^\circ$$

$$\therefore \qquad \delta = 45^\circ$$

**Case III :** Good conductors. For a good conductor,

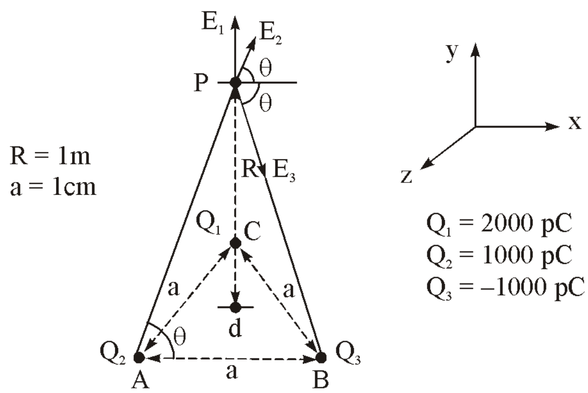
$$\vec{J}_C >> \vec{J}_D \text{ and } \sigma >> \omega\varepsilon$$

***Example 10 :*** Determine the electric field intensity at the point P for the arrangement shown in Fig.




***Solution :***

The configuration of the charges is shown in the figure, where the points A, B and C are marked.



$\vec{E}_1$ at P due to $Q_1$ at point C is given by

$$\vec{E}_1 = \frac{Q_1}{4\pi\varepsilon_0 (CP)^2} \text{ along } \overrightarrow{CP}$$

As R >> a, CP $\cong$ R

$$\therefore \qquad \vec{E}_1 = \frac{2000\times10^{-12}\times36\pi\times10^9}{4\pi(1)^2}\vec{a}_y$$

$$= 18\,\vec{a}_y \text{ V/m} \qquad ...(1)$$

$\vec{E}_2$ at 'P' due to $Q_2$ at A is given by

$$\vec{E}_2 = \frac{Q_2}{4\pi\varepsilon_0 (AP)^2} \text{ along } \overrightarrow{AP}$$

As AP $\approx$ R

$$\vec{E}_2 = \frac{1000\times10^{-12}\times36\pi\times10^9}{4\pi(1)^2}$$

$$= 9 \text{ V/m along } \overrightarrow{AP}$$

$$= 9\cos(\theta)\,\vec{a}_x + 9\sin(\theta)\vec{a}_y \qquad ...(2)$$

where 'θ' is shown in the Fig.

$\vec{E}_3$ at 'P' due to $Q_3$ at B is given by

$$\vec{E}_3 = \frac{Q_3}{4\pi\varepsilon_0 (BP)^2} \text{ along } \overrightarrow{BP}$$

As BP $\approx$ R

$$\vec{E}_3 = \frac{-1000\times10^{-12}\times36\pi\times10^9}{4\pi(1)^2}$$

$$= -9\text{V/m along } \overrightarrow{BP}$$

$$= 9 \text{ V/m along } \overrightarrow{PB}$$

$$= 9\cos(\theta)\vec{a}_x - 9\sin(\theta)\,\vec{a}_y \qquad ...(3)$$

$\therefore$ The resultant $\vec{E}$ field at 'P' is given by

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

From equations (1), (2) and (3)

$$\vec{E} = 18\,\vec{a}_y + 9\cos(\theta)\,\vec{a}_x + 9\sin(\theta)\,\vec{a}_y + 9\cos(\theta)\,\vec{a}_x - 9\sin(\theta)\,\vec{a}_y$$

$$= 18\cos(\theta)\vec{a}_x + 18\vec{a}_y$$

From the figure

$$\tan\theta \cong \frac{R}{\left(\frac{a}{2}\right)} = 200$$

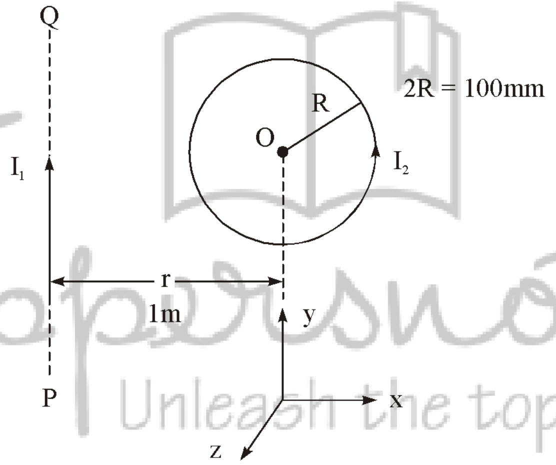
$$\therefore \qquad \cos\theta = \frac{1}{200}$$

$$\vec{E} = \left(0.09\,\vec{a}_x + 18\,\vec{a}_y\right)$$

***Example 11 :*** A infinitely long straight wire carries 1000 A of current and in the vicinity there is a circular conducting loop of 100 mm diameter with the centre of the loop 1 m away from the straight conductor. Both the wire and the loop are coplanar. Determine the magnitude and direction of current in the loop that procedures a zero flux density at its centre.

***Solution :***

The infinitely long wire PQ with $I_1$ = 1000 A and circular loop with center, O at r = 1m is shown in Fig.




Magnetic flux density, $B_1$ at O due to PQ is given by

$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi r}(-\hat{a}_z)$$

Let $I_2$ be the current through the circular loop such that $\vec{B}_2$ produced by it at O cancels with $\vec{B}_1$, giving zero resultant flux density, $\vec{B}$.

$\vec{B}_2$ is given by

$$\vec{B}_2 = \frac{\mu_0 I_2}{2R}\hat{a}_x$$

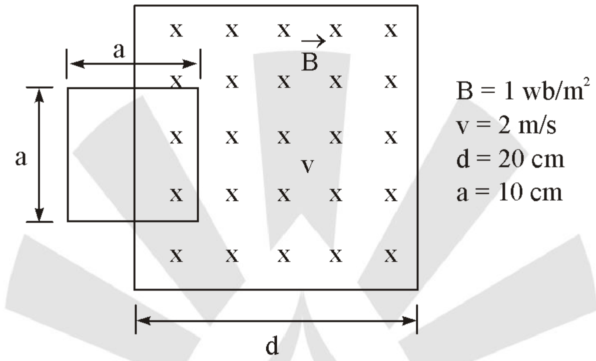
$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \left(\frac{\mu_0 I_2}{2R} - \frac{\mu_0 I_1}{2\pi r}\right)\hat{a}_z$$

$$\therefore \qquad I_2 = \frac{I_1 R}{\pi r} = \frac{I_1}{\pi}\frac{R}{r}$$

$$= \frac{1000}{\pi}\frac{50}{1000} A = 15.92 A$$

The direction of $I_2$ is counter clockwise.

***Example 12 :*** A square coil of 10 turns and 10 cm side is moved through a steady magnetic field of 1 Wb/$m^2$ at a constant velocity of 2 m/sec with its plane perpendicular to the field as shown in Fig. Plot the variation of induced e.m.f as the coil moves along the field.




***Solution :***

The square coil ABCD moving into the magnetic field is shown in Fig. 1

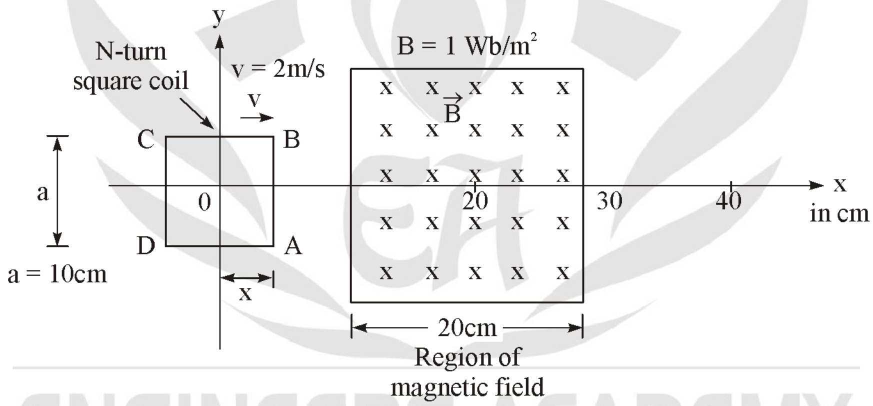



**Fig. (a)**

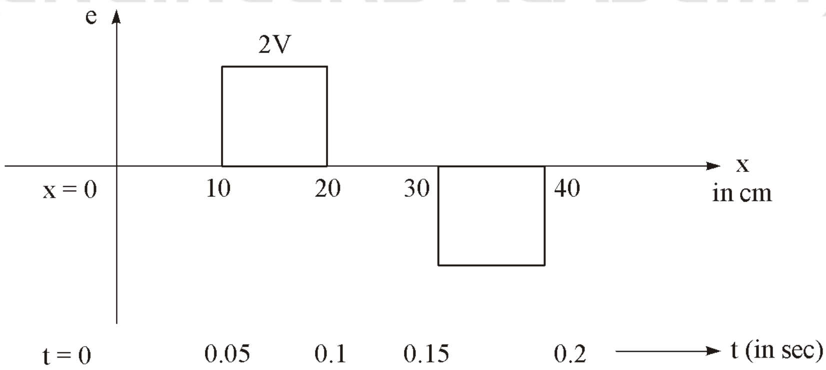



**Fig. (b)**

Let the right side of the coil AB be at x = 0 initially at t = 0.

As the coil moves in the positive direction of x, it comes under the influence of $\vec{B}$ (magnetic field) when x > 10 cm (t > 0.05 sec) and $\phi$ cut by the coil increases.

$\therefore$ $$\text{emf induced} = e = N\frac{d\phi}{dt} = N\frac{d\phi}{dx}\frac{dx}{dt}$$

where $d\phi = (a\,dx)\,B$

and $\frac{dx}{dt} = v$

$$e = N\,a\,B\,v = 10\times10^{-1}\times1\times2V = 2V$$

$\therefore$ For $10 < x < 20$ or $0.05 < t < 0.10$,

$$e = 2V$$

At t = 0.1 sec, the coil occupies the position $10 < x < 20$ where the flux cut by the coil reaches its maximum value $\phi_{max}$.

For $20 < x < 30$ or for $0.1 < t < 0.15$, $\phi$ does not change.

$\therefore$ $$\text{emf} = 0$$

For $30 < x < 40$, or $0.15 < t < 0.2$, $\phi$ decreases with time.

$\therefore$ $$e = -2\text{ V}$$

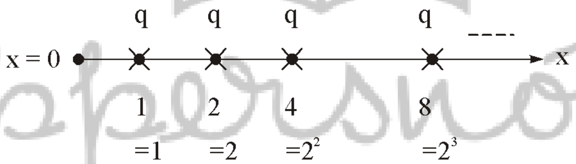
For $40 < x$, or $0.20 < t$, $\phi = 0$,

$\therefore$ $$e = 0$$

Variation of induced emf, e v/s x and e v/s t is shown in Fig. (b).

***Example 13 :*** An infinite number of charges, each equal to 'q' are placed along the x = 1, x = 2, x = 4, x = 8, x = 16 and so on. Find the potential and electric field at point x = 0, due to these system of charges.

***Solution :***




$$V = \frac{q}{4\pi\varepsilon}\left[\frac{1}{1}+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+....\right]$$

The infinite number of terms in the above summation are in geometric series.

$\therefore$ $$V = \frac{q}{4\pi\varepsilon}\left[\frac{\text{First term}}{1-\text{Common ratio}}\right] = \frac{q}{4\pi\varepsilon}\frac{1}{1-\frac{1}{2}} = \frac{q}{2\pi\varepsilon}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon}\left[\frac{1}{1}+\frac{1}{(2)^2}+\frac{1}{(2)^4}+\frac{1}{(2)^6}+...\right](-\vec{a}_x)$$

$$= (-\vec{a}_x)\left[\frac{\text{First term}}{1-\text{Common ratio}}\right]\frac{q}{4\pi\varepsilon}$$

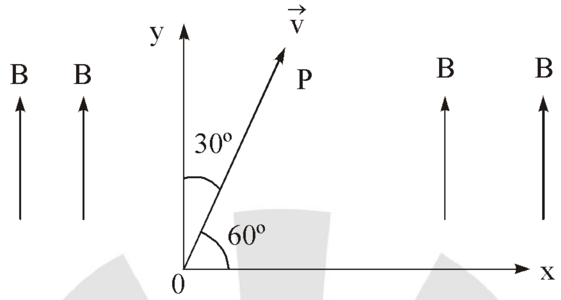
$$= \frac{q}{4\pi\varepsilon}\frac{1}{\left(1-\frac{1}{4}\right)} = \frac{q}{3\pi\varepsilon}$$

where $\vec{a}_x$ is the unit vector in the positive x-direction and $\varepsilon$ is the permittivity of the medium.

***Example 14 :*** An electron moves in the X-Y plane with a speed of $10^6$ m/s. Its velocity vector makes an angle of 60° with X axis. A magnetic field of magnitude $10^{-2}$ T exists along the Y axis. Compute the magnetic force exerted on the electron and its direction.

***Solution :***

Given : $\vec{v}$ = 10 m/s along $\overrightarrow{OP}$ as shown in Fig.




$$\vec{B} = \vec{a}_y \, 10^{-2} \text{T} \left(\text{or Wb/m}^2\right)$$

Magnetic force on the electron

$$= -e\left(\vec{v} \times \vec{B}\right) = -e\left(vB\sin\theta\right)\vec{a}_z$$

$$= -e\, 10^6\, 10^{-2} \sin(30°)\vec{a}_z$$

$$= -\frac{e}{2} 10^4 \, \vec{a}_z \text{ Newtons}$$

***Example 15 :*** A charge +Q is uniformly distributed throughout the volume of a dielectric sphere of radius 'R' and dielectric constant '$\varepsilon_R$' . Based on Gauss law, determine the expressions for the electric field, E as a function of distance 'r' from the center of the sphere, within the ranges 0 < r < R and R < r. Indicate expression (s) for the critical point (s) on the sketch.

***Solution :***

The charge distribution is shown in Fig. (1)

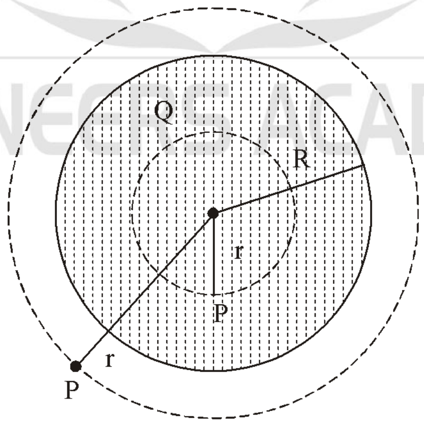



**Fig. (1)**

$$\text{Charge per unit volume} = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$$

For $0 < r < R$, when point P is inside the sphere, assume a spherical Gaussian surface through P. Applying Gauss law,

$$\oint \vec{D}\cdot d\vec{s} = \text{Charge enclosed}$$

$$D4\pi r^2 = \frac{3Q}{4\pi R^3}\cdot\frac{4}{3}\pi r^3$$

$$D4\pi r^2 = Q\frac{r^3}{R^3}$$

$$E = \frac{D}{\varepsilon} = Q\frac{r^3}{R^3\varepsilon_0\varepsilon_R 4\pi r^2}$$

$$E = \frac{Q}{4\pi R^3\varepsilon_0\varepsilon_R}r = K_1 r \qquad ...(1)$$

where, $K_1 = \dfrac{Q}{4\pi R^3\varepsilon_0\varepsilon_R}$

As $r \to R$, $E_1 = \dfrac{Q}{4\pi\varepsilon_0\varepsilon_R R^2}$

The sketch of E with r is a straight line passing through the origin. For $R < r < \infty$, $\varepsilon = \varepsilon_0$ (for air) and point P is outside the sphere,

$$D4\pi r^2 = Q$$

$$E = \frac{Q}{4\pi\,\varepsilon_0}\frac{1}{r^2} = \frac{K_2}{r^2} \qquad ...(2)$$

where, $K_2 = \dfrac{Q}{4\pi\varepsilon_0}$

As $r \to R$, $E_2 = \dfrac{Q}{4\pi\varepsilon_0 R^2}$

$$E_1 < E_2 \text{ as } \varepsilon_R > 1$$

The value of E rises from $E_1$ to $E_2$ along AB at the interface between the two medium $(r = R)$

The resultant sketch of E as a function of r is shown in Fig. 2

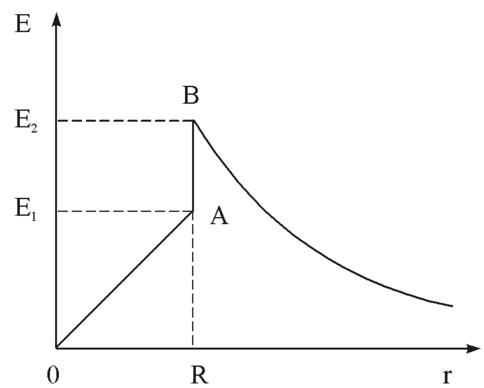



**Fig. (2)**

With critical points A and B at r = R

i.e., the interface between the two media.

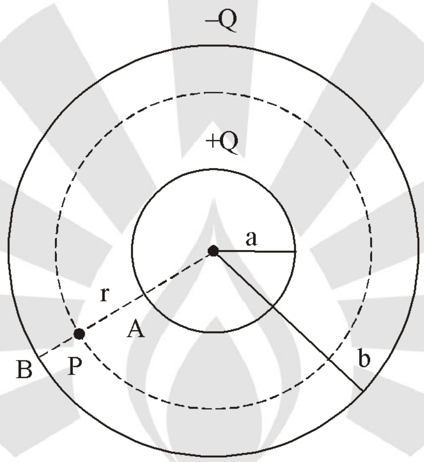
***Example 16 :*** Show via the construction of a suitable Gaussian surface, that the capacitance of a spherical capacitor consisting of two concentric shells of radii a and b is given by

$$C = 4\pi\varepsilon_0 \frac{ab}{(b-a)}$$

where $\varepsilon_0$ is the free space permittivity.

***Solution :***

The electric field intensity at a general point, P as shown in figure in the region between the concentric spherical shells, can be found by assuming a spherical Gaussian surface through point P. Let Q be the charge on the spherical shells. Applying Gauss law:




$$\oint \vec{D}\cdot d\vec{s} = Q$$

$$D\, 4\pi r^2 = Q$$

$$D = \frac{Q}{4\pi r^2} \text{ along } \hat{a}_r$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r,\; a < r < b$$

Potential Difference between the conducting shells

$$V_{AB} = -\int_B^A \vec{E}\cdot\vec{dL}$$

For $\vec{dL} = dr\,\vec{a}_r$

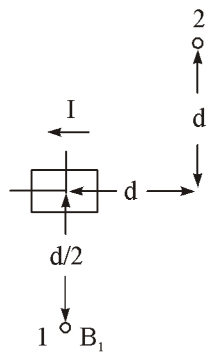
$$V_{AB} = -\int_B^A \frac{Q}{4\pi\varepsilon r^2} dr = \frac{Q}{4\pi\varepsilon r}\Big|_b^a$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C = \frac{Q}{V_{AB}} = \frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)}$$

$$= 4\pi\varepsilon\frac{ab}{(b-a)}$$

***Example 17 :*** A current I in the short conducting element shown in figure produces a flux density $B_1$ at point 1. Determine the magnitude and the direction of the flux density vector at point 2.




***Solution :***

The elemental (or differential) length dL through which I is flowing is shown in Fig. 1 The Magnetic flux density at any point P due to a differential current element 1 $\overrightarrow{dL}$ is given by

$$\vec{B} = \frac{\mu I\,\overrightarrow{dL}\times\hat{a}_R}{4\pi R^2}$$

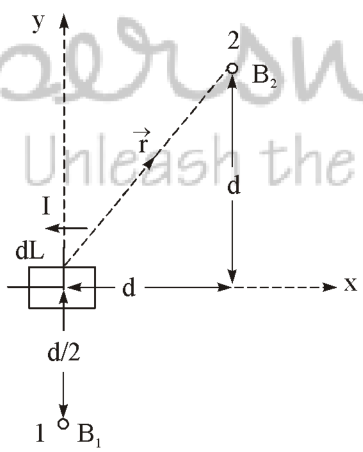



**Fig. (1)**

where R is the distance from the center and $\hat{a}_R$ is the unit vector along the line from the element to the point.

$\therefore$ $\vec{B}_1$ at point 1 is given by

$$\vec{B}_1 = \frac{\mu I dL\sin(90^o)}{4\pi(d/2)^2}\vec{a}_z$$

$$= \frac{\mu \text{ I dL}}{\pi d^2} \quad ...(1)$$

$\therefore$ $\vec{B}$ at point 2 is given by

$$\vec{B}_2 = \frac{\mu \text{ I } \overrightarrow{dL} \times \hat{a}_r}{4\pi r^2}$$

where, $r^2 = d^2 + d^2 = 2d^2$

and $\overrightarrow{dL} \times \hat{a}_r = dL \sin(180 + 45)\hat{a}_y$

$$= -\frac{dL}{\sqrt{2}}\hat{a}_z$$

$$\therefore \quad \vec{B}_2 = \frac{\mu I\left(-\frac{dL}{\sqrt{2}}\hat{a}_z\right)}{4\pi(2d^2)}$$

$$= -\frac{\mu I \text{ dL}}{8\sqrt{2}\ \pi d^2}\hat{a}_z \quad ...(2)$$

$$\therefore \quad \vec{B}_2 = -\frac{1}{8\sqrt{2}}\vec{B}_1$$

***Example 18 :*** Given the potential function V = 2x + 4y in free space, find stored energy in a 1– $m^3$ volume centered at origin.

***Solution :***

Electric field intensity is $E = -\nabla V = -\left(\frac{\partial v}{\partial x}a_x + \frac{\partial v}{\partial y}a_y + \frac{\partial v}{\partial z}a_z\right)$

$$E = -2a_x - 4a_y \text{ (V/m)}$$

$$|E| = \sqrt{20}\text{V / m}$$

Energy density in space is given by

$$W_E = \frac{1}{2}\in E^2$$

So energy stored in 1 – $m^3$ Volume is

$$W = \frac{1}{2}\in E^2 \times 1$$

$$= \frac{1}{2} \times \frac{1}{36\pi} \times 10^{-9} \times 20$$

$$= \frac{10^{-8}}{36\pi}\text{J}$$

***Example 19 :*** For an electric field $E = E_0 \sin \omega t$, what is the phase difference between the conduction current and the displacement current?

***Solution :***

The conduction current is defined as

$$J_c = \sigma E = \sigma E_0 \sin \omega t$$

where, $\sigma$ is conductivity and the displacement current density is

$$J_d = \frac{\partial D}{\partial t} = \in \frac{\partial E}{\partial t}$$

$$= \in \omega E_0 \sin\left(\frac{\pi}{2} - \omega t\right)$$

So the phase difference between $J_c$ and $J_d$ is $90^o$.

***Example 20 :*** A circular conducting loop of radius 2 m, centered at origin in the plane z = 0 carries a current of 4A in the $a_\phi$ direction, What will be the magnetic field intensity at origin ?

***Solution :***

According to Biot-savart law, magnetic field intensity any point P due to the current element $Idl$ is defined as

$$H = \int \frac{idl \times R}{4\pi R^3}$$

where, R is the vector distance of point P from the current element.

Here current is flowing in $a_\phi$ direction.

So the small current element

$$Idl = I\rho d\phi a_\phi$$

$$= 4 \times 2 d\phi a_\phi = 8 d\phi a_\phi$$

and since the magnetic field to be determined at center of the loop so we have

$$R = 2m \qquad \text{(radius = 2m)}$$

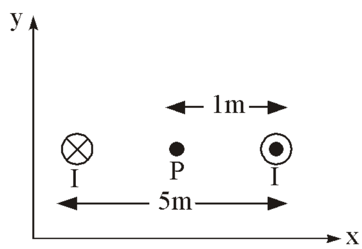
$$a_R = -a_\rho \qquad \text{(pointing towards origin)}$$

Therefore the magnetic field intensity at origin is

$$H = \int_0^{2\pi} \frac{(8 d\phi a_\phi) \times (-a_\rho)}{4\pi (2)^2}$$

$$= \int_0^{2\pi} \frac{8}{16\pi} d\phi a_z = \frac{a_z}{2\pi}[\phi]_0^{2\pi} = a_z \text{ A/m}$$

***Example 21 :*** Two infinitely long wires separated by a distance 5 m, carry currents I in opposite direction as shown in the figure



If I = 8 A, then the magnetic field intensity at point P is

***Solution :***

The magnetic field intensity produced at a distance ρ from an infinitely long straight wire carrying current I is defined as

$$H = \frac{I}{2\pi\rho}$$

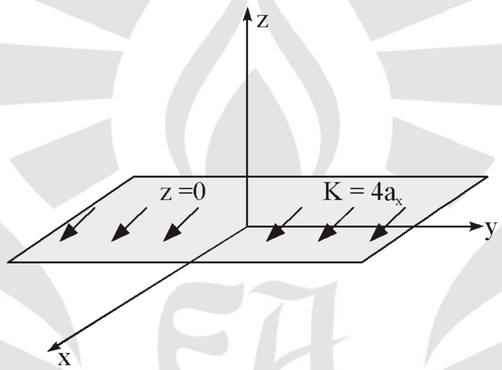
As determined by right hand rule, the direction of magnetic field intensity will be same (in $-a_y$ direction) due to both the current source. So, at point P the net magnetic field intensity due to both the current carrying wire will be

$$H = H_1 + H_2$$

$$= \frac{I}{2\pi(4)}(-a_y) + \frac{1}{2\pi(1)}(-a_y)$$

$$= -\frac{5(8)}{8\pi}a_y = -\frac{5}{\pi}a_y \qquad (I = 8A)$$

***Example 22 :*** An infinite current sheet with uniform surface current density $K = 4a_x$ A/m is located at z = 0 as shown in figure

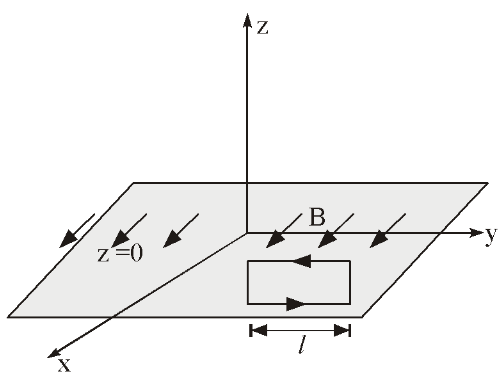



Magnetic flux density at any point above the current sheet (z > 0) will be

***Solution :***

For determining the magnetic field at any point above the plane z = 0, we draw a rectangular Amperian loop parallel to the y-z plane and extending an equal distance above and below the surface as shown in the figure.

From Ampere's circuital law, $\int B.dl = \mu_0 I_{enc}$



Since the infinite current sheet is located in the plane

$$z = 0$$

So, the z-component of the magnetic flux density will be cancelled due to symmetry and in the closed Amperian loop the integral will be only along y-axis. Thus we have

$$B(2l) = \mu_0 I_{enc}$$

$$2Bl = \mu_0 K l \qquad (I_{enc} = Kl)$$

As determined by right hand rule, the magnetic flux density above the plane z = 0 will be in $-a_y$ direction. So we have the flux density above the current sheet as

$$B = -\frac{\mu_0 \times 4}{2} a_y$$

$$= -2\mu_0 a_y \text{wb/m}^2 \qquad (K = 4 \text{ A/m})$$

***Alternate Method :*** The magnetic flux density produced at any point P due to an infinite sheet carrying uniform current density K is defined as

$$B = \frac{1}{2}\mu_0 (K \times a_n)$$

where, $a_n$ is the unit vector normal to the sheet directed toward the point P.

So, magnetic flux density at any point above the current sheet $K = 4a_x$ is

$$B = \frac{1}{2}\mu_0 (4\alpha_x) \times (a_z)$$

$$= 2\mu_0 a_y \text{ wb/m}^2 \qquad (a_n = a_z)$$

***Example 23 :*** Vector magnetic potential in a certain region of free space is $A = (6y - 2z)\, a_x + 4xza_y$. The electric current density at any point (x, y, z) will be

***Solution :***

The magnetic field intensity, (H) in the terms of magnetic vector potential, (A) is defined as

$$H = \frac{1}{\mu_0}(\nabla \times A)$$

$$= \frac{1}{\mu_0}[\nabla \times (6y - 2z)a_x + 4xza_y]$$

$$= \frac{1}{\mu_0}[-8a_x - 2a_y + 6a_z]$$

Since the electric current density at any point is equal to the curl of magnetic field intensity at that point.

i.e., $$J = \nabla \times H$$

So, we have the electric current density in the free space as

$$J = \nabla \times \frac{1}{\mu}[-8a_x - 2a_y + 6a_z] = 0$$

❍❍❍