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ENGINEERS ACADEMY

# RPSC - A.En.

## Assistant Engineering

## ELECTRICAL

Rajasthan Public Service Commission (RPSC)

## Volume - 8

## Control System

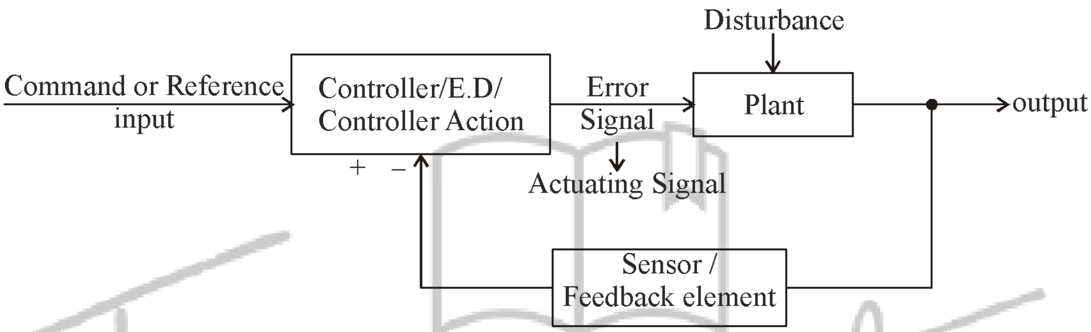
SIERRA INNOVATIONS

# INTRODUCTION TO CONTROL SYSTEM

## THEORY

## 1.1 Introduction

A **control system** is a system of devices or set of devices, that manages, commands, directs or regulates the behaviour of other device(s) or system(s) to achieve desire results. In other words the **definition of control system** can be rewritten as **A control system is a system, which controls other system**.




The objective of any control system is to ensure that controlled output is same as the command (desired) or not.

## 1.2 Basic Terminologies

**Controlled Variable and Control Signal or Manipulated Variable :** The controlled variable is the quantity or condition that is measured and controlled. The control signal or manipulated variable is the quantity or condition that is varied by the controller so as to affect the value of the controlled variable. Normally, the controlled variable is the output of the system.

**Plant :** A plant may be a piece of equipment, perhaps just a set of machine parts functioning together, the purpose of which is to perform a particular operation. Any physical object to be controlled (such as a mechanical device, a heating furnace, a chemical reactor, or a spacecraft) a plant.

**Processes :** A process to be a natural, progressively continuing operation or development marked by a series of gradual changes that succeed one another in a relatively fixed way and lead toward a particular result or end; or an artificial or voluntary, progressively continuing operation that consists of a series of controlled actions or movements systematically directed toward a particular result or end.

**Disturbances :** A disturbance is a signal that tends to adversely affect the value of the output of a system. If a disturbance is generated within the system, it is called internal, while an external disturbance is generated outside the system and is an input.

**Controllers :** A controller is one which compares controlled values with the desired values and has a function to correct the deviation produced.

Controller performs two works :

(i) Error detection

(ii) Control action

When feedback is present then controller performs the error detection and controlled action.

When feedback is not present then only controlled action is done.

## 1.3 Feature of Control System

The main feature of control system is

(i) There should be a clear mathematical relation between input and output of the system.

(ii) When the relation between input and output of the system can be represented by a linear proportionality, the system is called linear control system.

(iii) Again when the relation between input and output cannot be represented by single linear proportionality, rather the input and output are related by some non-linear relation, the system is referred as non-linear control system.

## 1.4 Requirement of Good Control System

**Accuracy :** Accuracy is the measurement tolerance of the instrument and defines the limits of the errors made when the instrument is used in normal operating conditions. Accuracy can be improved by using feedback elements. To increase accuracy of any control system error detector should be present in control system.

**Sensitivity :** The parameters of control system are always changing with change in surrounding conditions, internal disturbance or any other parameters. This change can be expressed in terms of sensitivity. Any control system should be insensitive to such parameters but sensitive to input signals only.

**Noise :** An undesired input signal is known as noise. A good control system should be able to reduce the noise effect for better performance.

**Stability :** It is an important characteristic of control system. For the bounded input signal, the output must be bounded and if input is zero then output must be zero then such a control system is said to be stable system.

**Bandwidth :** An operating frequency range decides the bandwidth of control system. Bandwidth should be large as possible for frequency response of good control system.

**Speed :** It is the time taken by control system to achieve its stable output. A good control system possesses high speed. The transient period for such system is very small.

**Oscillation :** A small numbers of oscillation or constant oscillation of output tend to system to be stable.

## 1.6 Signal

**(i) Physical definition :** It is form of energy that contains information of some phenomenon in a pattern.

**(ii) Mathematical definition :** It is a function of one or more no of independent variables.

We will always consider a signal as

**f(t)** → One independent variable that is time **'t'**.

or **f(x, t)** → two independent variables that is time **'t'** and distance **'x'**.

But function may be independent of time i.e.,

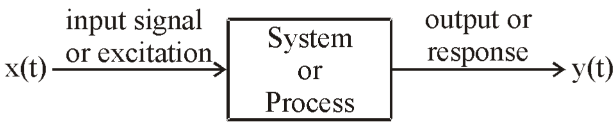
$$\mathbf{f(t)} = \text{constant function}$$

## 1.7 Systems

A system is a combination of components that act together and perform a certain objective. A system need not be physical.

**Representation of System :**

**1. Graphical Representation of System :**




In control system analysis we are not bothered or think that is inside a system, we only wants to know the behaviour of the system.

(i) Block-diagram representation

(ii) Signal flow graph representation

(iii) Electrical network representation

(iv) Mechanical Diagram representation

**2. Mathematical Representation :**

(i) Algebraic equation representation

**Ex.:** $y(t) = kx(t) + C$

(ii) Differential equation representation

**Ex.:** $\frac{dy(t)}{dt} + y(t) = x(t)$

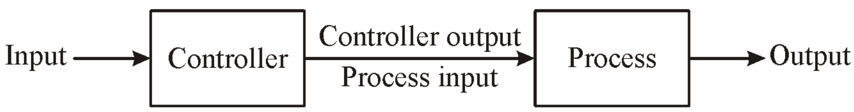
(iii) Difference equation representation

**Ex.:** $y(t) + y(t - 1) = x(t + 2)$

***Note :*** *To convert differential and difference equation in linear we use Laplace transform and Z-transform respectively.*

## 1.7 Classification of System

**1. Open Loop System**




- This is the system in which the output of the system is function of input only.
- Present output in such a system depends upon present input only.
- There are absence of error detecting element.

**Practical example**

- **Electric Hand Drier :** Hot air (output) comes out as long as you keep your hand under the machine, irrespective of how much your hand is dried.
- **Automatic Washing Machine :** This machine runs according to the preset time irrespective of washing is completed or not.
- **Bread Toaster :** This machine runs as per adjusted time irrespective of toasting is completed or not.

- **Automatic Tea/Coffee Maker :** These machines also function for pre adjusted time only.
- **Timer Based Clothes Drier :** This machine dries wet clothes for pre – adjusted time, it does not matter how much the clothes are dried.
- **Light Switch :** Lamps glow whenever light switch is on irrespective of light is required or not.
- **Volume on Stereo System :** Volume is adjusted manually irrespective of output volume level.

**Advantages of Open Loop Control System**

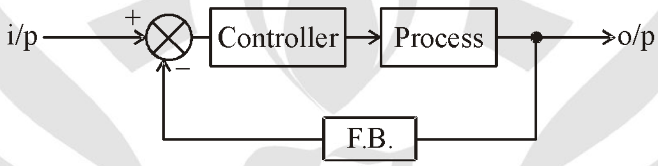
- Simple in construction and design.
- Economical.
- Easy to maintain.
- Generally stable.
- Convenient to use as output is difficult to measure.

**Disadvantages of Open Loop Control System**

- They are unreliable.
- Any change in output cannot be corrected automatically.
- Effect of non linearity is more.
- This systems are inaccurate.
- Output affected by parameter variation.
- Output affected by noise variation.

**2. Close Loop System**

These are the system in which present output is function of present input as will as previous output.




**Practical Example**

- **Automatic Electric Iron :** Heating elements are controlled by output temperature of the iron.
- **Servo Voltage Stabilizer :** Voltage controller operates depending upon output voltage of the system.
- **Water Level Controller :** Input water is controlled by water level of the reservoir.
- **Missile Launched & Auto Tracked by Radar :** The direction of missile is controlled by comparing the target and position of the missile.
- **An Air Conditioner :** An air conditioner functions depending upon the temperature of the room.
- **Cooling System in Car :** It operates depending upon the temperature which it controls.

**Advantages of Closed Loop Control System**

- Closed loop control systems are more accurate even in the presence of non-linearity.
- Highly accurate as any error arising is corrected due to presence of feedback signal.
- Bandwidth range is large.
- Facilitates automation.
- The sensitivity of system may be made small to make system more stable.
- This system is less affected by noise.
- Accurate and reliable.

**Disadvantages of Closed Loop Control System**

- They are costlier.
- They are complicated to design.
- Required more maintenance.
- Feedback leads to oscillatory response.
- Overall gain is reduced due to presence of feedback.
- Stability is the major problem and more care is needed to design a stable closed loop system.

**Feedback Loop of Control System**

A feedback is a common and powerful tool when designing a control system.

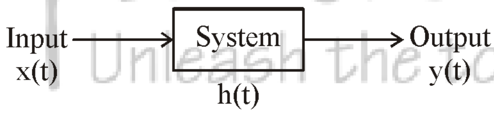
When feedback signal is positive then system called positive feedback system. For positive feedback system, the error signal is the addition of reference input signal and feedback signal. When feedback signal is negative then system is called negative feedback system. For negative feedback system, the error signal is given by difference of reference input signal and feedback signal.

**Effect of Feedback**

- Error between system input and system output is reduced.
- System gain is reduced by a factor $1/(1 \pm GH)$.
- Improvement in sensitivity.
- Stability may be affected.
- Improve the speed of response.

## 1.8 Transfer Function

Transfer function of Linear Time Invariant system is defined as the ratio of Laplace transform of output to the Laplace transform of input under a zero initial condition.




$$\text{Output} = f\text{ (input and system)}$$

$$y(t) = f(x(t) \text{ and } h(t))$$

For Liner Time Invariant system $\quad y(t) = x(t) \otimes h(t)$

Laplace transform $\quad Y(s) = X(s)H(s)$

$$\frac{Y(s)}{X(s)} = H(s) = \text{Transfer function (only for Linear Time Invariant system)}$$

To analyses the characteristics of the system mathematical the system represented in either of the two standard model, manually.

(i) Transfer function model

(ii) State model

***Note :*** *We study the state model in details in state space analysis.*

## Initial Conditions

Let $y(t) = kx(t) + c$

where, $x(t)$ = input

$y(t)$ = output, k and c are constant

If $x(t) = 0$

Still $y(t) = c \neq 0$, (Initial condition)

To find out the transfer function of Linear Time Invariant system we have to ignored the initial condition i.e., zero.

Hence, $$\left.\frac{y(t)}{x(t)}\right|_{c=0} = k$$

In control system we study only Linear Time Invariant system. Here we consider three variables as

$t \rightarrow$ time (sec)

$\omega \rightarrow$ frequency (rad/sec)

$s \rightarrow$ complex frequency

$s = \sigma + j\omega \Rightarrow$ Gives time characteristics (known as damping factor or convergence.)

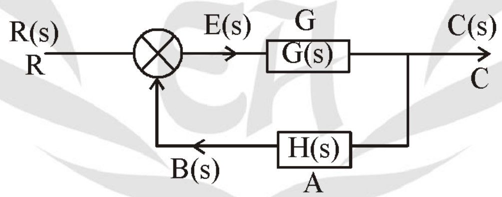
$s = j\omega \Rightarrow$ gives frequency factor or factor.

**Note :** $s = \sigma + j\omega$,

$\sigma$ will dump the effect of $\omega$ and thus $\sigma$ kills the frequency $\omega$.

## Feedback Topology

- $R(s) \rightarrow$ Reference input
- $C(s) \rightarrow$ Controlled output
- $B(s) \rightarrow$ Feedback signal
- $E(s) \rightarrow$ Error signal




Here $$G(s) = \frac{C(s)}{E(s)} \rightarrow \text{Forward path transfer function}$$

$$H(s) = \frac{B(s)}{C(s)} \rightarrow \text{Feedback path transfer function}$$

$$E(s) = R(s) \pm B(s)$$

$$G(s)H(s) = \frac{B(s)}{E(s)}$$

**Case I :** If feedback path is disconnected there E(s) becomes R(s).

Thus $$G(s)H(s) = \frac{B(s)}{R(s)}$$

The term G(s)H(s) is know as open loop transfer function.

**Case II :** If feed back is presents

Then $\frac{C(s)}{R(s)}$ is close loop transfer function.

Now, $C(s) = G(s)E(s) = G(s)(R(s) \pm B(s)) = G(s)\ (R(s) \pm C(s)H(s))$

$$C(s)(1 \pm G(s)H(s)) = R(s)G(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$ → For negative feedback system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$ → For positive feedback system.

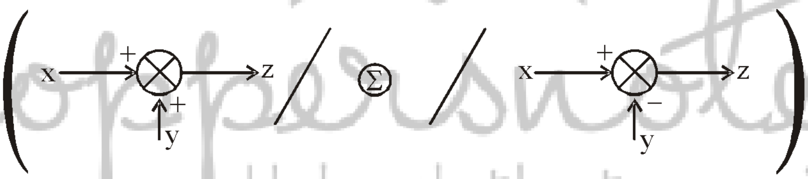
## 1.9 Block Diagram

The block diagram is to represent a control system in diagram form.

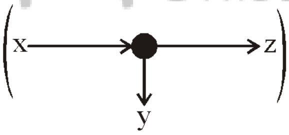
**Elements of Block Diagram Reduction (Three-Elements)**

(i) Block (G)

(ii) Adder or summer point

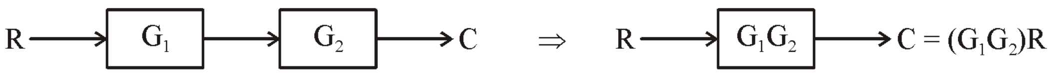



(iii) Take off point

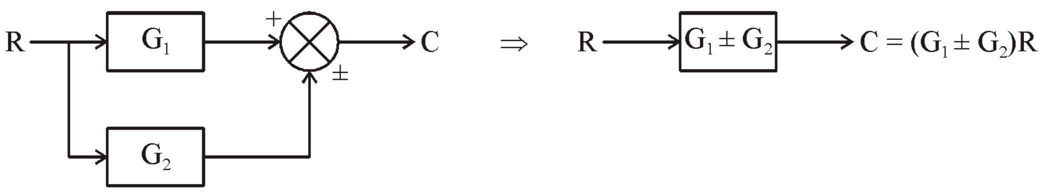



**Block diagram reduction rules :**

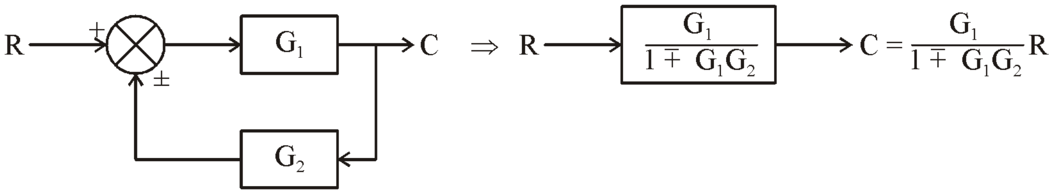
**1.** Combining blocks in cascade



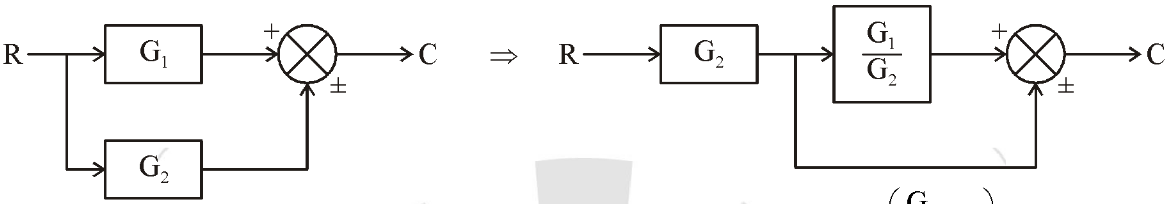

**2.** Combining blocks in parallel or eliminating a forward loop



**3.** Eliminating a feedback loop

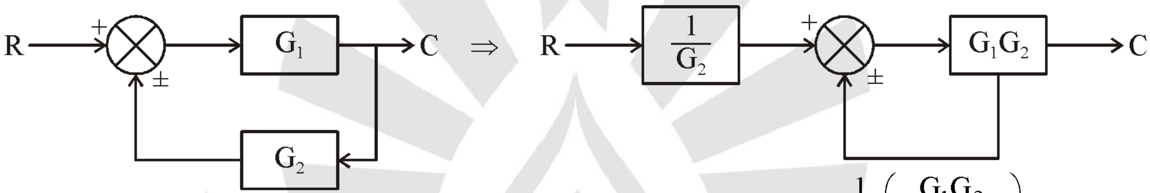



**4.** Removing a block from a forward path



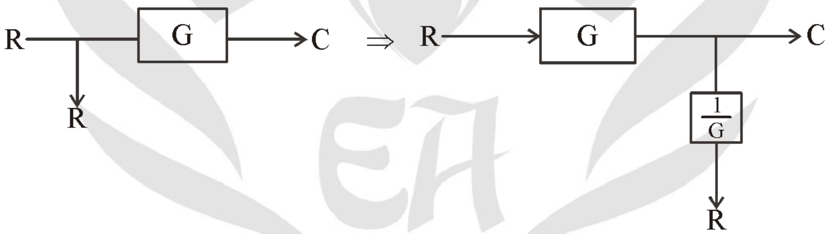

$$C = G_2\left(\frac{G_1}{G_2} \pm 1\right)R$$

**5.** Removing a block from a feedback loop

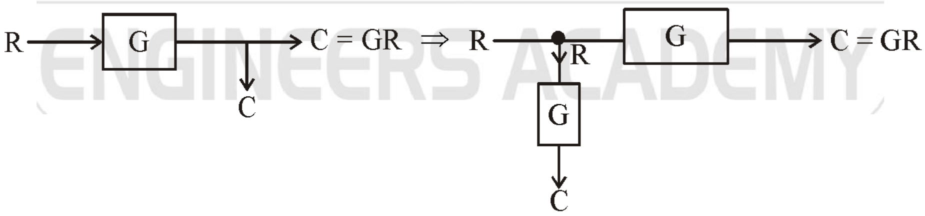



$$C = \frac{1}{G_2}\left(\frac{G_1G_2}{1 \mp G_1G_2}\right)R$$

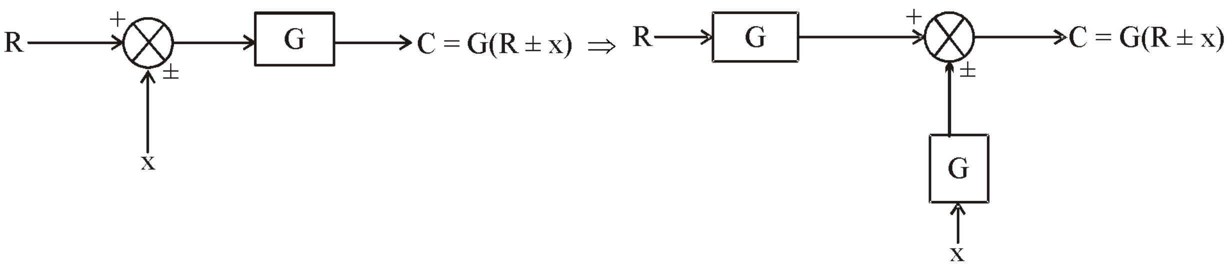
**6.** Moving a take-off point beyond a block



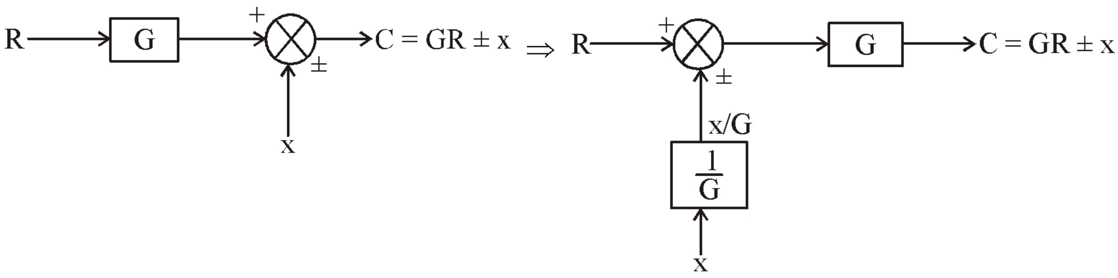

**7.** Moving a take-off point ahead of a block



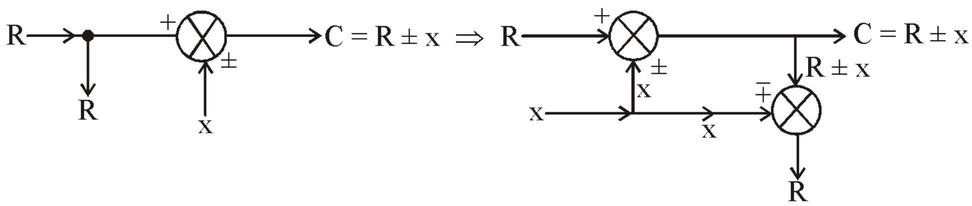

**8.** Moving a summing point beyond a block



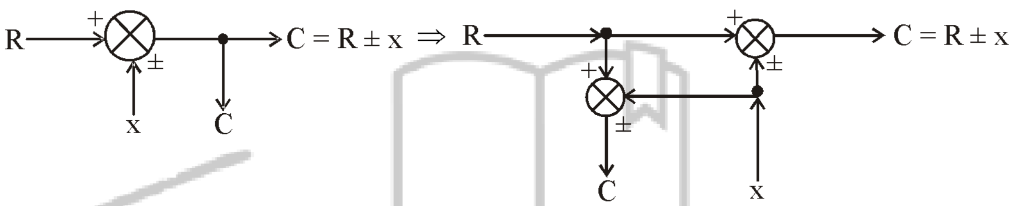
**9.** Moving a summing point ahead of a block



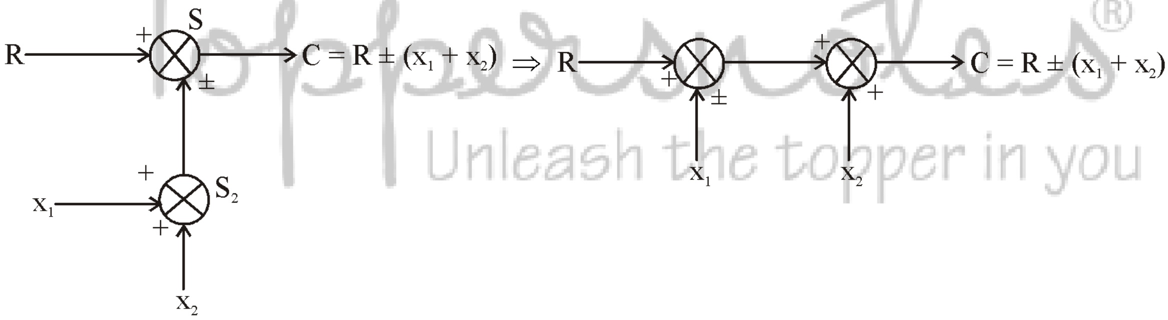

**10.** Moving a take-off point beyond a summing point



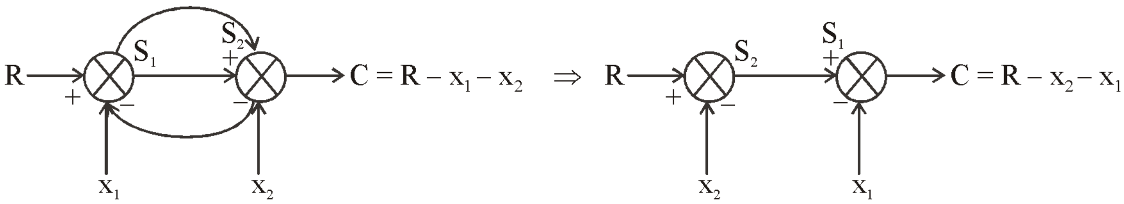

**11.** Moving a take-off point ahead of a summing point




**12.** Rearranging summing points




**13.** Rearranging summing points

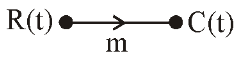





## 1.10 Signal Flow Graph (By S.J. Mason)

Signal flow graph is a graphical technique that deals with the relation between variables of system that are described in the form of a set of linear algebraic equations.

**Ex.:**

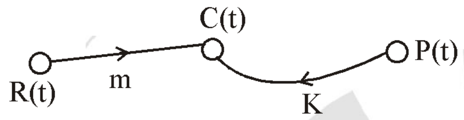
(i) $C(t) = m.R(t) \Rightarrow$ 


$C(t)$ → output, $R(t)$ → input

(ii) $C(t) = mR(t) + k,\ k \neq 0$

For this equation k can not be represented in signal flow graph because it does not contain any variables.

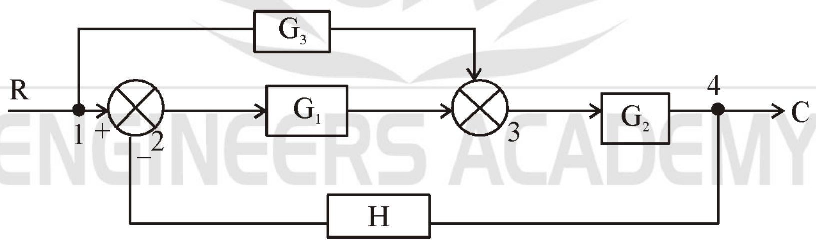
(iii) $C(t) = mR(t) + kP(t)$



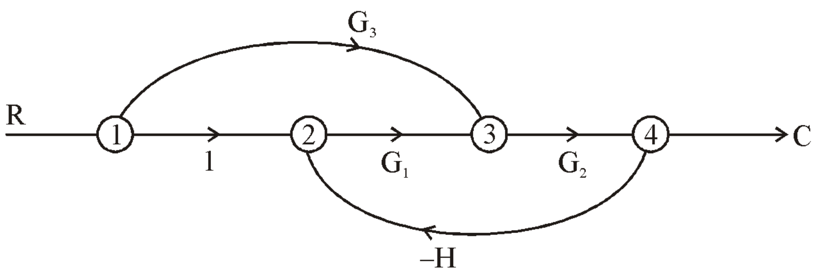

**Definitions for signal flow graph**

1. **Node :** A node in the signal flow graph represents a variables of the system. Number of nodes is equal to the number of variables.
   - **(a) Input Node :** The node that has only out going branch.
   - **(b) Output Node :** The node that has only incoming branch.
2. **Forward Path :** The path that connect to the input node to the output node such that no any node is repeated.
3. **Loop :** A loop is a closed path that starts and ends at same node such that number node is repeated.
4. **Non Touching Loop :** If two or more loops have no any common node then they called non touching loop.
5. **Forward path gain :** It is the product of branch gains encountered in traversing a forward path.
6. **Loop gain :** It is the product of branch gains encountered in traversing a forward path.

**To convert block diagram into signal flow graph**




**Note :** Number of nodes = sum of (Number of take off point + Number of adder)




Here number of node = 4

**To convert equation into signal flow graph**

$$x = x_1 + B_3u$$

$$\dot{x}_1 = -a_1x_1 + x_2 + B_2u$$

$$\dot{x}_2 = -a_2x_1 + B_1u$$

Convert this equation into algebraic equation by taking Laplace transform

$$X(s) = X_1(s) + B_3U(s)$$

$$sX_1(s) = -a_1X_1(s) + X_2(s) + B_2U(s)$$

$$sX_2(s) = -a_2X_1(s) + B_1u(s)$$

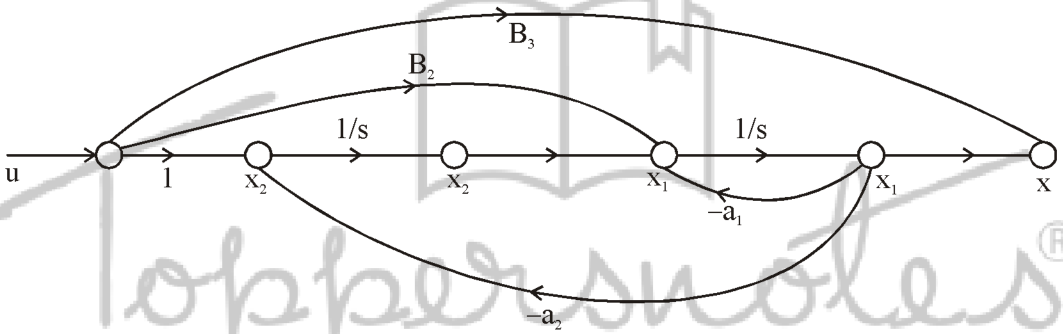
**Note :** With the help of signal flow graph. We find transfer function of system thus we consider initial condition zero. If we take Laplace transform of differential equation.

$$\frac{dx}{dt} = \dot{x}$$

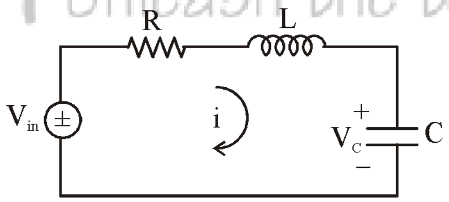
$\Rightarrow$ Laplace transform $(\dot{x}) = sX(s) - X(0)$

Here $X(0) = 0$ (initial condition)

For given differential equation $x \to$ output and $u \to$ input.




**To convert electrical network into signal flow graph**

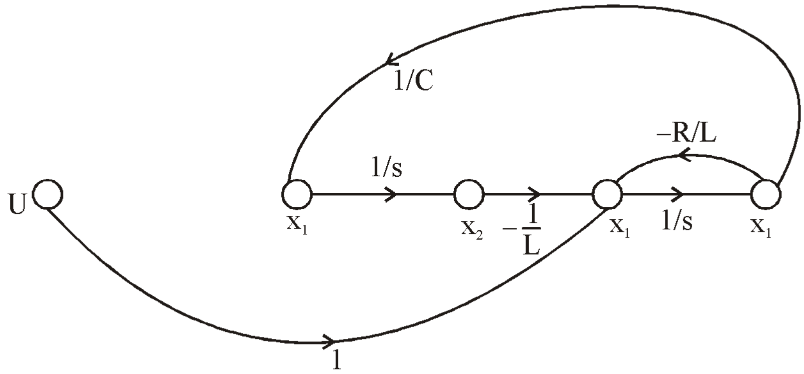



$$V_{in} = Ri + \frac{Ldi}{dt} + V_c$$

$$\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{L}V_c + \frac{1}{L}V_{in}$$

$$i_c = c\frac{dV_c}{dt}$$

$$\frac{dV_c}{dt} = \frac{i_c}{C}$$




But $i \to x_1$

$v_c \to x_2$

$v_{in} \to x$

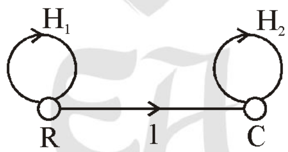
$$\dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}U$$

$$\dot{x}_2 = \frac{1}{C}x_1 + 0.x_2 + 0.U$$

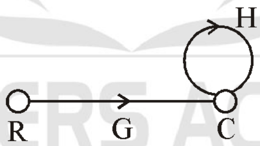
## Mason's gain formula

This formula is used to find transfer function of system, which is described in signal flow graph. This is applicable in input output node only

$$\text{Transfer function} = \sum_{k=1}^{n} \frac{p_k \Delta_k}{\Delta}, \; k = 1, 2, ....n$$




We can not apply mason's gain formula because here R is not incoming branch, $H_1$ is an incoming branch.




We can apply Mason's gain formula because R is incoming branch

$\Rightarrow$ $$\text{Transfer Function} = \sum_{k=1}^{n} \frac{p_k \Delta_k}{\Delta}$$

$n \to$ Number of forward path

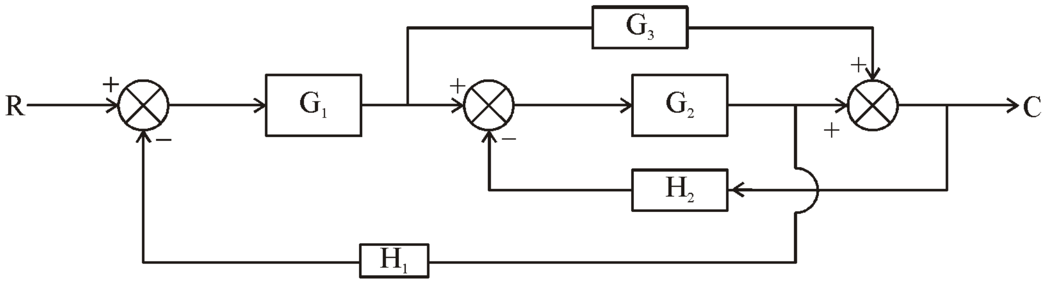
$p_k \to$ gain of $k^{th}$ forward path

$\Delta \to$ Determinant of graph

$\Delta$ = 1 – [sum of all individual "loop gain"] + [some of product of 2 non-touching "loop gain"] – [sum of product of 3-non touching "loop gain"] + [ ....

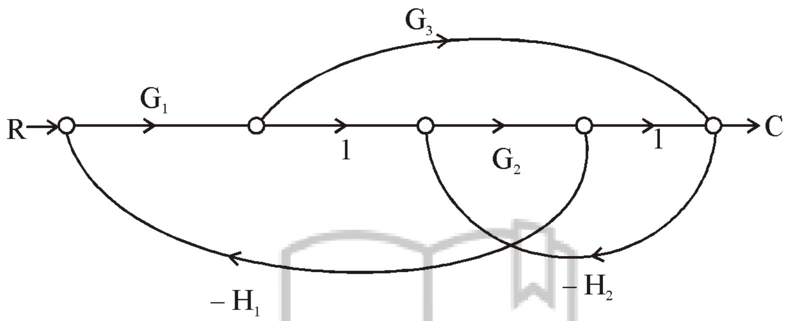
$\Delta_k$ = 1 – [sum of all individual "loop gain" that are not touching to $k^n$ forward path] + [sum of product of 2-non-touching "loop gain" that are not touching to $k^n$ forward path] – [......

*Example 1 :* Find $\frac{C}{R}$ of given block diagram




***Solution :***

**By signal flow graph**




Forward Path $P_1 = G_1G_2$

$P_2 = G_1G_3$

Loop $L_1 = -\ G_1G_2H_1$

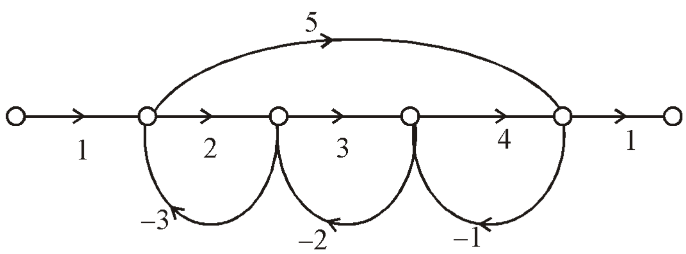
$L_2 = -\ G_2H_2$

$L_3 = G_1G_3\ (-H_2)\ G_2(-H_1)$

$= G_1G_2G_3H_1H_2$

$$\frac{C}{R} = \frac{G_1G_2[1-0]+G_1G_3[1-0]}{1+G_1G_2H_1+G_2H_2-G_1G_2G_3H_1H_2}$$

*Example 2 :* Find $\frac{C}{R}$ for the figure given below




***Solution :***

Forward path = 2 $P_1 = 24$

$P_2 = 5$

Loop = 4

$L_1 = 2 \times -3 = -6$

$L_2 = 3 \times -2 = -6$

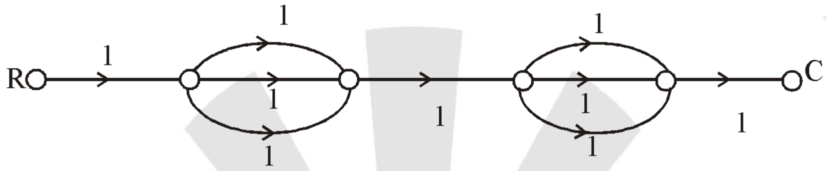
$L_3 = 4 \times -1 = -4$

$L_4 = 5 \times -1 \times -2 \times -3 = -30$

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{24[1-0]+5[1-(-6)]}{1-[(-6)+(-6)+(-4)+(-30)]+[(-6)\times(-4)]} = \frac{59}{71}$$

***Example 3 :*** Find $\frac{C}{R}$ for the figure given below



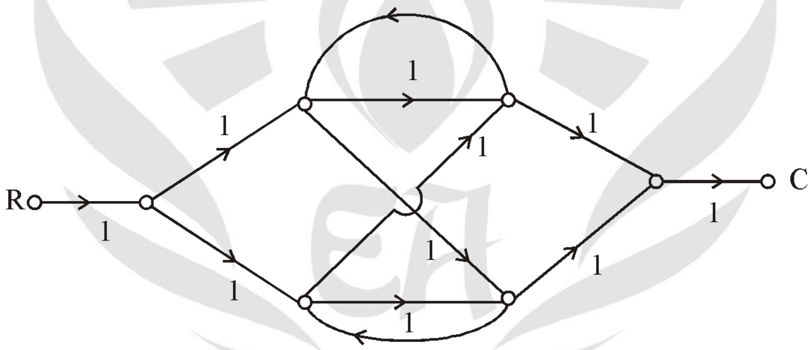

***Solution :***

Forward path = 9

Loop = 0

$$\frac{C}{R} = \frac{1+1+\ldots 9\text{ times}}{1-[0]} = 9$$

***Example 4 :*** Find number of forward paths and number of loops.

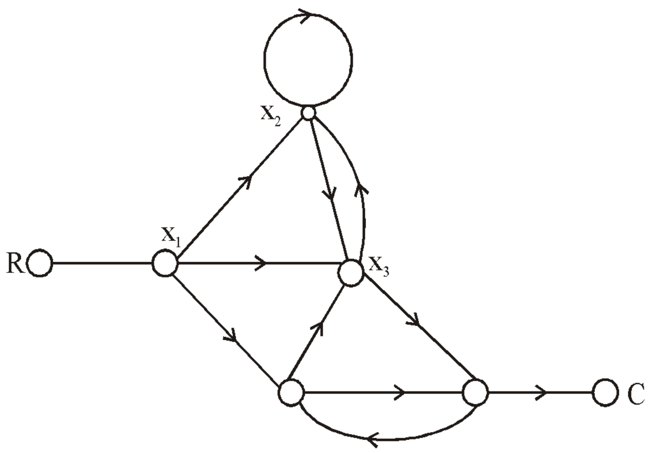



***Solution :***

Forward path = 6

Loop = 3

***Example 5 :*** Find number of forward paths and number of loops.

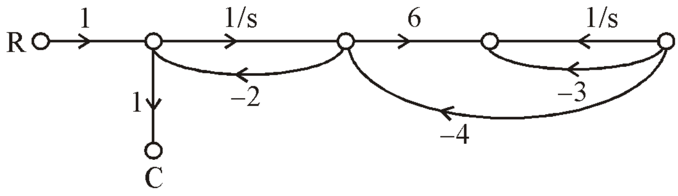


***Solution :***

Forward path = 4

Loop = 4

***Example 6 :*** Find $\frac{C}{R}$ for given signal flow graph



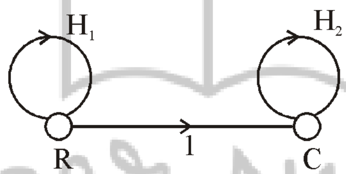

***Solution :***

Forward path = 1

Loop = 3

$$\frac{C}{R} = \frac{1\left[1-\left\{-\frac{24}{s}-\frac{3}{s}\right\}+0\right]}{1-\left[-\frac{2}{s}-\frac{24}{s}-\frac{3}{s}\right]+\frac{6}{s^2}} = \frac{s(s+27)}{s^2+29s+6}$$

***Example 7 :*** Find $\frac{C}{R}$ for given signal flow graph




***Solution :***

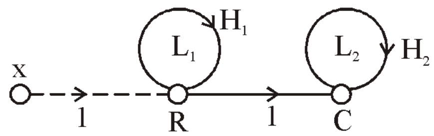
Here Mason's gain formula can not apply, because R is not incoming branch, $H_1$ is incoming to R. C is output node; all branches are incoming. Then we solve as

$$C = R \times 1 + CH_2$$

$$\Rightarrow \quad \frac{C}{R} = \frac{1}{1-H_2}$$

But if we want to do it by mason's gain formula

Consider a node also with branch gain 1.

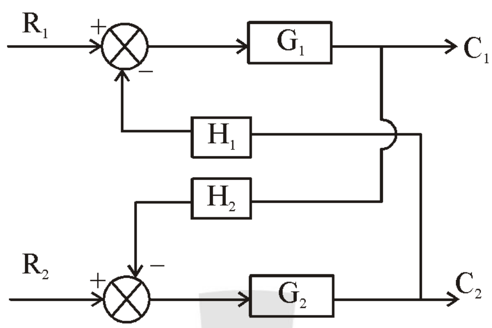



$$\frac{C}{R} = \frac{C/X}{R/X} = \frac{\frac{1[1-0]}{\Delta}}{\frac{1[1-H_2]}{\Delta}} = \frac{1}{1-H_2}$$

$$\Delta = 1 - [H_1 + H_2]$$

**Multi input multi output system**

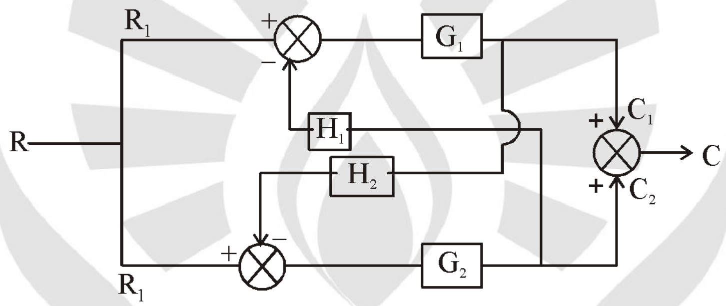
***Example 8 :*** Find $\frac{C}{R}$ for the figure given below




***Solution :***

$$C = C_1 + C_2$$
$$R_1 = R_2 = R$$

We get




Here $C_1$ and $C_2$ are affected by both $R_1$ and $R_2$. Thus,
We get 4 T/F.

$$\frac{C_1}{R_1} = K_1, \frac{C_2}{R_2} = K_2$$

$$\frac{C_2}{R_1} = K_3, \frac{C_1}{R_2} = K_4$$

$$C = C_1 + C_2 \text{ and } C_1 = K_1R_1 + K_2R_2,\ C_2 = K_3R_1 + K_4R_2$$

$$\therefore \quad \frac{C}{R} = k_1 + k_2 + k_3 + k_4$$

$$\Rightarrow \quad \frac{C_1}{R_1} = \frac{G_1}{1 - G_1G_2H_1H_2}, \qquad \frac{C_1}{R_2} = \frac{-G_1G_2H_1}{1 - G_1G_2H_1H_2}$$

$$\frac{C_2}{R_1} = \frac{-G_1G_2H_2}{1 - G_1G_2H_1H_2}, \qquad \frac{C_2}{R_2} = \frac{G_2}{1 - G_1G_2H_1H_2}$$

$$\therefore \quad \frac{C}{R} = \frac{G_1 + G_2 - G_1G_2H_1 - G_1G_2H_2 + 0}{1 - G_1G_2H_1H_2}$$

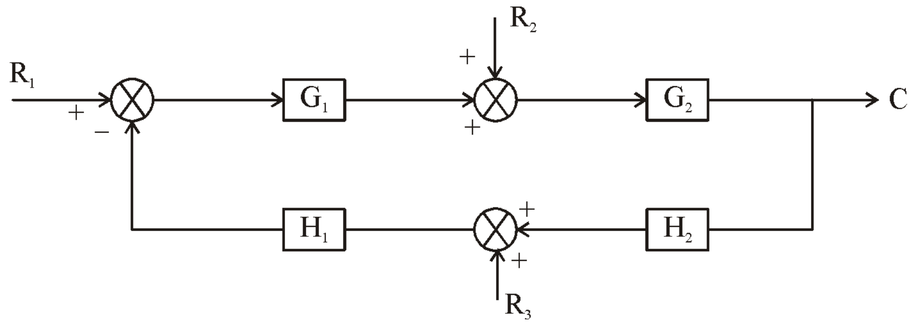
**Note :** Standard close loop

$$\text{Transfer function} = \frac{G(s)}{1 \pm G(s)H(s)}$$

$\Rightarrow$ $q(s) \rightarrow$ characteristics equation.

Roots of $q(s) = 1 \pm G(s)H(s)$, give closed loop poles

***Example 9 :*** Find $\frac{C}{R}$ for the figure given below




Find output of system.

***Solution :***

$$C = CR_1 + CR_2 + CR_3$$

$\Rightarrow$ $CR_1$ when $R_2 = R_3 = 0$

$$\frac{C}{R_1} = \frac{G_1G_2}{1+G_1G_2H_1H_2} \Rightarrow C = \frac{G_1G_2R_1}{1+G_1G_2H_1H_2}$$

Similarly,

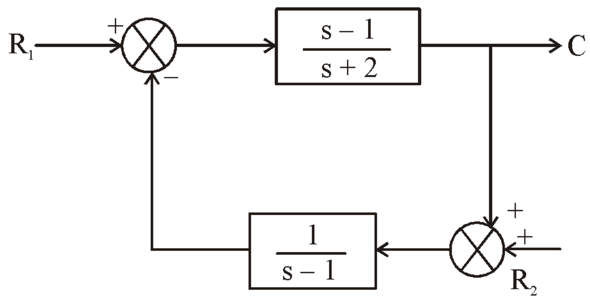
$$\frac{C}{R_2} = \frac{G_2}{1+G_1G_2H_1H_2} \Rightarrow C = \frac{G_1R_2}{1+G_1G_2H_1H_2}$$

$$\frac{C}{R_3} = \frac{-H_1G_1G_2}{1+G_1G_2H_1H_2} \Rightarrow C = \frac{-G_1G_2H_1R_3}{1+G_1G_2H_1H_2}$$

$\therefore$

$$C = \frac{G_1G_2R_1 + G_2R_2 - G_1G_2H_1R_3}{1+G_1G_2H_1H_2}$$

***Example 10 :*** Find the characteristics equations



***Solution :***

For $R_2 = 0$

$$\frac{C}{R_1} = \frac{s-1/s+2}{1+\left(\frac{s-1}{s+2}\right)\frac{1}{(s-1)}} = \frac{s-1}{s+3}$$

For $R_1 = 0$

$$\frac{C}{R_2} = \frac{-\left(\frac{1}{s-1}\right)\left(\frac{s-1}{s+2}\right)}{1+\left(\frac{s-1}{s+2}\right)\left(\frac{1}{s-1}\right)} = \frac{-1}{s+3}$$

Characteristics equations $q(s) = s + 3$

## 1.11 Mathematical Modeling

Representation of a system with the help of a differential equation is called mathematical model of a system.

**Classification of a System on The Basis of Their Mathematical Model :**

**1. Linear Time Invariant System :** These are the system which are represented by linear differential equation with coefficient which are independent of time.

**Ex.:** $$a_0\frac{d^n y(t)}{dt^n} + a_1\frac{dy^{n-1}(t)}{dt^{n-1}} + a_2\frac{d^{n-2}y(t)}{dt^{n-2}} + \ldots + a_n = x(t)$$

$y(t) \rightarrow$ output and $x(t) \rightarrow$ input.

$a_0$, $a_1$ .... $a_n$ are coefficient of independent of time.

**2. Linear Time Variant system :** These are the system in which, the linear differential equation representing, the system has coefficient which are function of time.

**Ex.:** $$(\sin 2t)\frac{d^2y(t)}{dt^2} + t^2\frac{dy}{dt} + 5y(t) = x(t)$$

**3. Non Linear Time Invariant System :** These are the system which are represented by non linear differential equation with coefficient which are independent of time.

**Ex.:** $$2\frac{d^2y(t)}{dt^2} + 5\left(\frac{dy(t)}{dt}\right)^2 + 8y = x(t)$$

**4. Non Linear Time Variant System :** These are the system which are represented by non linear difference equation which has coefficient as function of time.

**Ex.:** $$\frac{d^3y(t)}{dt^3} + (\log t)\left(\frac{d^2y(t)}{dt^2}\right)^2 + 5\frac{dy(t)}{dt} + \log y(t) = x(t)$$

## Mathematical Modelling of Mechanical Systems

There are two types of mechanical systems.

Linear mechanical system

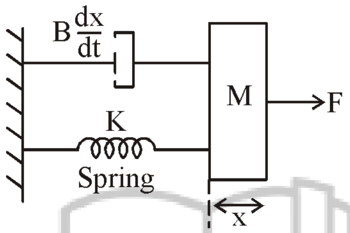
Rotational mechanical

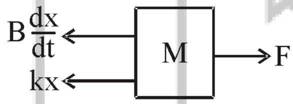
**The linear mechanical type of systems** have three variables

1. Force which is represented by 'F'
2. Velocity which is represented by 'V'
3. Linear displacement represented by 'X'

And also they have three parameters :

1. Mass which is represented by 'M'.
2. Coefficient of viscous friction which is represented by 'B'.
3. Spring constant which is represented by 'K'.




$$\Sigma F_x = M\frac{d^2x}{dt^2} = ma$$

$$F - B\frac{dx}{dt} - kx = M\frac{d^2x}{dt^2}$$

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = F;\ \text{Mass force equation.}$$

**The rotational mechanical type of systems** have three variables

1. Torque which is represented by 'T'.
2. Angular velocity which is represented by 'ω'.
3. Angular displacement represented by 'θ'.

And also they have two parameters

1. Moment of inertia which is represented by 'J'.
2. Coefficient of viscous friction which is represented by 'B'.

## Mathematical modeling of Electrical System

**The electrical type of systems** have three variables

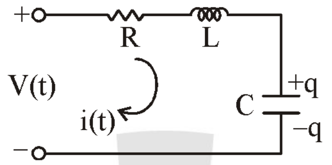
1. Voltage which is represented by 'V'.
2. Current which is represented by 'I'.
3. Charge which is represented by 'Q'.

And also they have three parameters which are active and passive elements

1. Resistance which is represented by 'R'.
2. Capacitance which is represented by 'C'.
3. Inductance which is represented by 'L'.

Electrical and mechanical types of systems have two types of analogies and they are written below

**Force voltage analogy :** In order to understand this type of analogy, let us consider a circuit which consists of series combination of resistor, inductor and capacitor.




$$V(t) = Ri(t) + \frac{Ldi(t)}{dt} + \frac{1}{C}\int i(t)dt$$

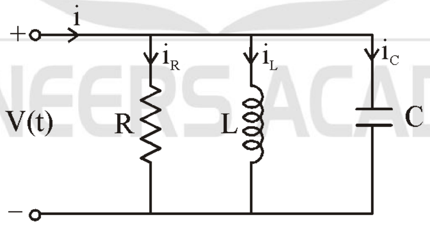
But $$i(t) = \frac{dq}{dt}$$

∴ $$\frac{Ld^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = v(t);\ \text{Called voltage equation.}$$

Now comparing the above with that derived for the mechanical system find that

1. Mass (M) is analogous to inductance (L).
2. Force is analogous to voltage V.
3. Displacement (x) is analogous to charge (Q).
4. Coefficient of friction (B) is analogous to resistance R.
5. Spring constant is analogous to inverse of the capacitor (C).

**Force Current Analogy :** In order to understand this type of analogy, let us consider a circuit which consists of parallel combination of resistor, inductor and capacitor.




$$i = i_R + i_L + i_C$$

$$i = \frac{V_R(t)}{R} + \frac{1}{L}\int V_L(t)dt + c\frac{dV_c(t)}{dt}$$

If $\phi$ is Magnetic Flux

$$V(t) = V_L(t) = V_C(t) = \frac{d\phi}{dt} \qquad \left[e = -\frac{d\phi}{dt}\right]$$

∴ $$i(t) = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi$$

## 1.12 Sensitivity and Feedback Characteristic

During the design process the engineers may want to consider the extent to which changes in system parameters affects the behaviour of a system. Ideally parameter changes due to heat or other causes should not appreciably affect a system's performance

The degree to which changes in system parameters affect system transfer function and hence performance is called "sensitivity".

The use of feedback in a control system reduces the effect of parameter variations.

### Sensitivity of system to their parameter variations

It is defined as the ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches to zero.

$$S_P^F = \lim_{\Delta P \to 0} \frac{\text{Fractional change in the function F}}{\text{Fractional change in the parameter P}}$$

$$S_P^F = \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P} = \lim_{\Delta P \to 0} \frac{P}{F} \cdot \frac{\Delta F}{\Delta P}$$

or $$S_P^F = \frac{\partial F / F}{\partial P / P} \qquad \frac{\partial F}{\partial P} \to \text{Partial differentiation of 'F' w.r.t. 'P'}$$

$$S_P^F = \frac{\partial F}{\partial P} \cdot \frac{P}{F} \qquad S_P^F \to \text{sensitivity of F w.r.t. 'P'}$$

**Note :** Sensitivity high ⇒ System is less stable

**Sensitivity of Open Loop System :**

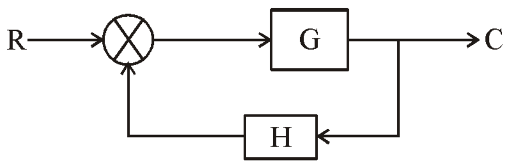

$$T = \frac{C}{R} = G$$

∴ $$C = GR \ (C \propto G)$$

∴ $$S_G^T = \frac{\partial T}{\partial G} \cdot \frac{G}{T} = 1 \times \frac{G}{G} = 1 \ \text{(100\%) sensitivity}$$

Thus open loop systems are highly sensitive w.r.t. parameter variation, because output of system changes directly to the parameter 'G'.

**Sensitivity of Close Loop System :**




$$T = \frac{C}{R} = \frac{G}{1+GH}$$

The transfer function 'T' is functions of both 'G' and 'H'. Thus two sensitivity are defined.

(i)
$$S_G^T = \frac{\partial T}{\partial G}\cdot\frac{G}{T}$$

$$S_G^T = \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right)\cdot\frac{G}{\left(\frac{G}{1+GH}\right)} = \frac{(1+GH)\cdot 1 - G(0+H)}{(1+GH)^2}\cdot(1+GH)$$

$$S_G^T = \frac{1}{1+GH}$$

Thus, $S_G^T \rightarrow$ reduces due to negative feedback with factor $\left(\frac{1}{1+GH}\right)$.

Hence sensitivity decrease.

(ii)
$$S_H^T = \frac{\partial T}{\partial H}\cdot\frac{H}{T}$$

$$S_H^T = \frac{\partial}{\partial H}\left(\frac{G}{1+GH}\right)\frac{H}{\left(\frac{G}{1+GH}\right)} = \frac{(1+GH)\cdot 0 - G(0+G)}{(1+GH)^2}\cdot\frac{H(1+GH)}{G}$$

$$S_H^T = \frac{-G^2\cdot H}{(1+GH)\cdot G}$$

$$S_H^T = \frac{-GH}{1+GH}$$

Thus, $S_G^T$ is increases on increasing the feedback parameter 'H'.

**Note :**

**1.** $S_G^T \rightarrow$ positive and $S_H^T \rightarrow$ negative. Thus they will works to cancel each other's effect.

**2.** Sensitivity for the feedback system w.r.t. both 'G' and 'H' are positive, which increases the sensitivity and hence stability of system is decreases.

**For Practical System we Assume :**

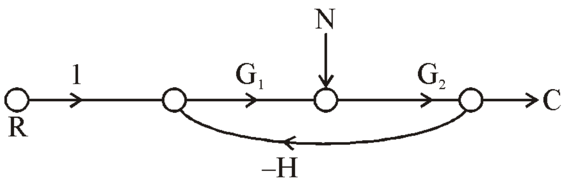
$$GH >> 1$$

**Note :** The term '(1 + GH)' is known as "Sensitive factor" or "Return difference".

## 1.13 Effect of Noise

Consider a system representing by signal flow graph as shown.

Let noise 'N' is added sin a system as



Thus the total output of the system becomes

$$C = C_R + C_N$$

$$C_R \text{ at } N = 0$$

$$\therefore \qquad \frac{C_R}{R} = \frac{G_1G_2}{1+G_1G_2H} \Rightarrow C_R = \left(\frac{G_1G_2}{1+G_1G_2H}\right)\cdot R$$

at $\qquad C_N$ at $R = 0$

$$\frac{C_N}{N} = \frac{G_2}{1+G_1G_2H}$$

$$C_N = \left(\frac{G_2}{1+G_1G_2H}\right)\cdot N$$

For practical system, $1 + G_1G_2H \simeq G_1G_2H$

$$\therefore \qquad C_R = \frac{1}{H}\cdot R \qquad ...(1)$$

$$C_N = \frac{1}{G_1H}\cdot N \qquad ...(2)$$

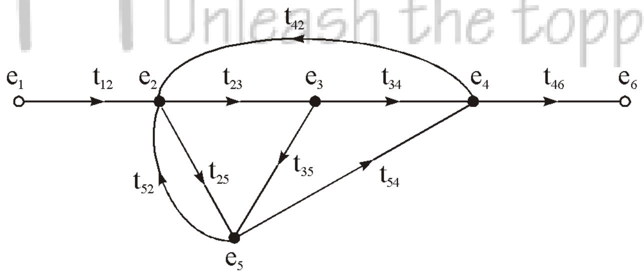
From equations (1) and (2) it is clear that effect of noise can be reduces by increasing '$G_1$' without effecting output of '$C_R$'.

If we increase 'H' then $C_N$ and $C_R$ both decreases.

If we increases only $G_1$ without affecting the feedback H only $C_N$ is decreases, which is desirable.

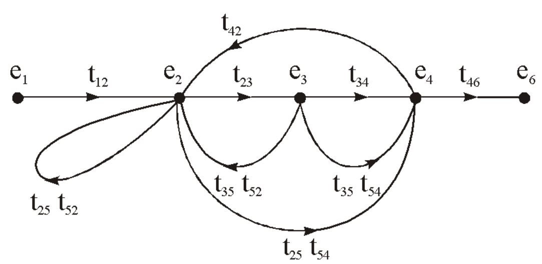
***Example 11 :*** Reduce the signal flow graph shown in figure below, to obtain another graph which does not contain the node $e_5$. (Also, remove any self-loop from the resulting graph)






***Solution :***

By eliminating the node $e_5$, the signal flow graph is shown below



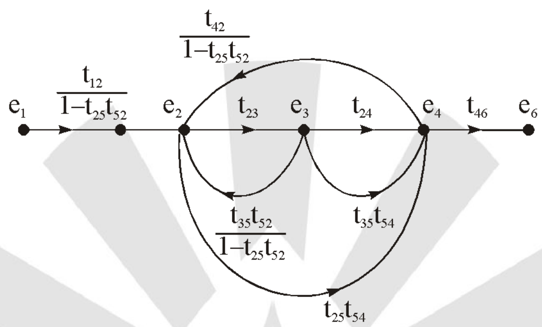
The Node equation

$$e_2 = e_1 t_{12} + e_2 t_{25} t_{52} + e_3 t_{35} t_{52} + e_4 t_{42}$$

$$e_2 (1 - t_{25} t_{52}) = e_1 t_{12} + e_3 t_{35} t_{52} + e_4 t_{42}$$

$$e_2 = e_1 \frac{t_{12}}{1 - t_{25} t_{52}} + e_3 \frac{t_{35} t_{52}}{1 - t_{25} t_{52}} + e_4 \frac{t_{42}}{1 - t_{25} t_{52}}$$

The redrawn signal flow graph is shown below



*Example 12 :* Draw a signal flow graph for the following set of algebraic equations:

$$y_2 = ay_1 - gy_3$$

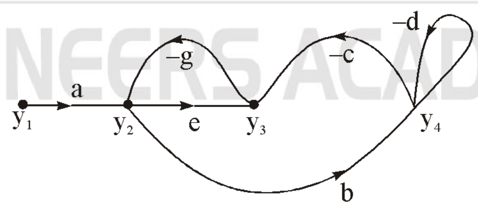
$$y_3 = ey_2 - cy_4$$

$$y_4 = by_2 - dy_4$$

Hence, find the gains $\frac{y_2}{y_1}$ and $\frac{y_3}{y_1}$

***Solution :***

The signal flow graph is drawn below

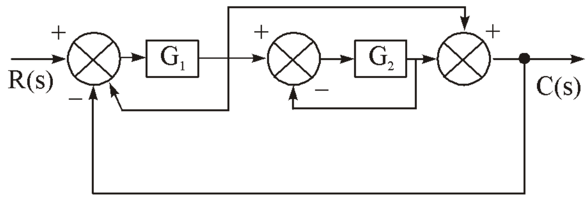


Applying the Mason's gain formula

$$\frac{y_2}{y_1} = \frac{a(1+d)}{1+d+eg+bcg+edg}$$

$$\frac{y_3}{y_1} = \frac{ae(1+d)+abc}{1+d+eg+bcg+edg}$$

***Example 13 :*** A feedback control system is shown in figure




(a) Draw the signal-flow graph that represents the system.

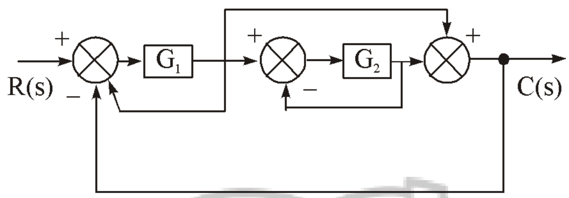
(b) Find the total number of loops in the graph and determine the loop-gains of all the loops.

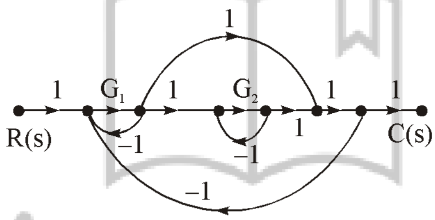
(c) Find the number of all possible combinations of non-touching loops taken two at a time.

(d) Determine the transfer function of the system using the signal-flow graph.

***Solution :***

(a) at every summing point and take off or pick off point consider nodes.




(b) Loops :

$$L_1 = -G_1$$
$$L_2 = -G_2$$
$$L_3 = -G_1$$
$$L_4 = -G_1G_2$$

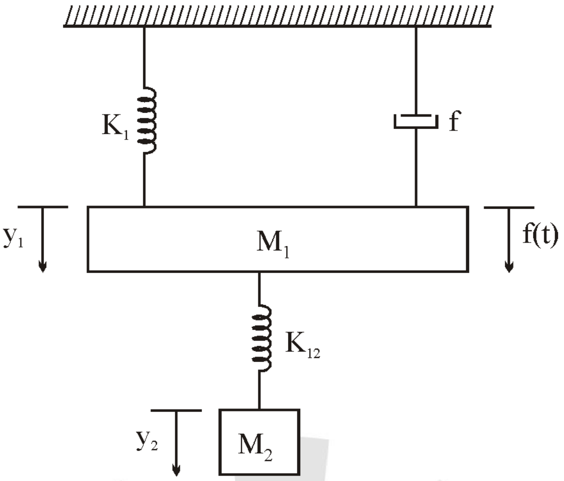
(c) Two Non Touching loops.

$$L_1L_2 = G_1G_2$$
$$L_2L_3 = G_1G_2$$

(d) $$\text{Transfer function} = \frac{G_1G_2 + G_1(1+G_2)}{1-(-G_1-G_2-G_1-G_1G_2)+G_1G_2+G_1G_2}$$

$$\frac{C(s)}{R(s)} = \frac{G_1G_2 + G_1(1+G_2)}{1+2G_1+G_2+3G_1G_2}$$

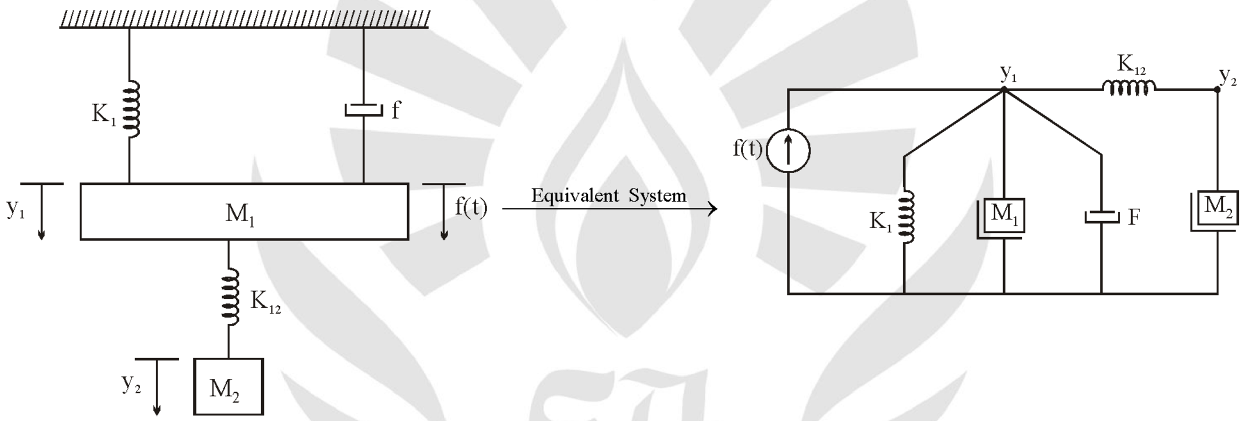
***Example 14 :*** A dynamic vibration absorber is shown in the above figure. The system is seen in many situations involving machines containing several unbalanced components. The parameters $M_2$ and $K_{12}$ may be chosen such that the main Mass $M_1$ does not vibrate when $F(t) = a \sin \omega_0 t$.




(a) Obtain the differential equation describing the system

(b) Draw the analogous electric circuit based on Force current analogy

***Solution :***




Differential equation

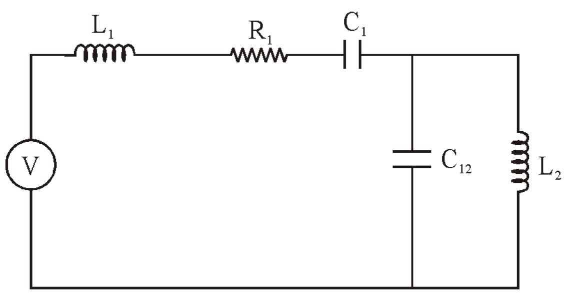
at node $y_1$, $$f(t) = M_1\frac{d^2y_1}{dt^2} + F\frac{dy_1}{dt} + k_1y_1 + k_{12}(y_1 - y_2)$$

at node $y_2$, $$0 = M_2\frac{d^2y_2}{dt^2} + k_{12}(y_2 - y_1)$$

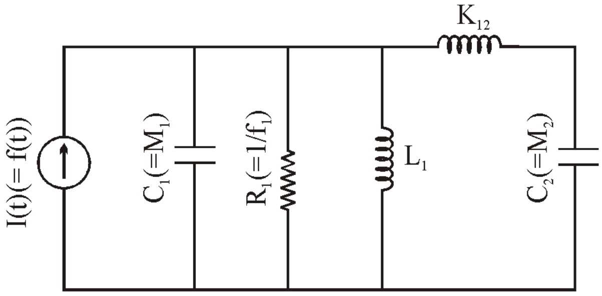


Equivalent electrical system

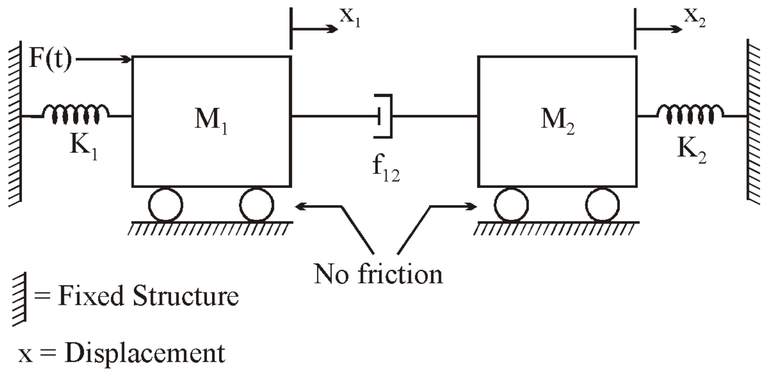
F-V analogy




F-I analolgy



***Example 15 :*** For the mechanical system of figure shown below



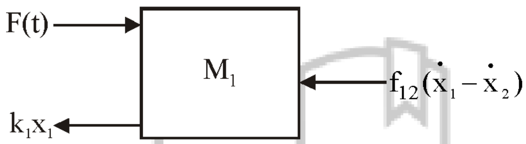

(a) Obtain the differential equations of the mechanical system.

(b) Sketch the mechanical equivalent representation.

(c) Draw the electrical analogous circuit based on force current analogy.

***Solution :***

Differential equation for mass $M_1$




Free Body Diagram of $M_1$

$$F(t) - k_1x_1 - f_{12}(\dot{x}_1 - \dot{x}_2) = M_1\ddot{x}_1$$

$$\Rightarrow \quad M_1\ddot{x}_1 + f_{12}(\dot{x}_1 - \dot{x}_2) + K_1x_1 = F(t) \qquad \text{...(i)}$$

Differential equation for $M_2$


Free Body Diagram of $M_2$

$$f_{12}(\dot{x}_1 - \dot{x}_2) - K_2x_2 = M_2\ddot{x}_2$$

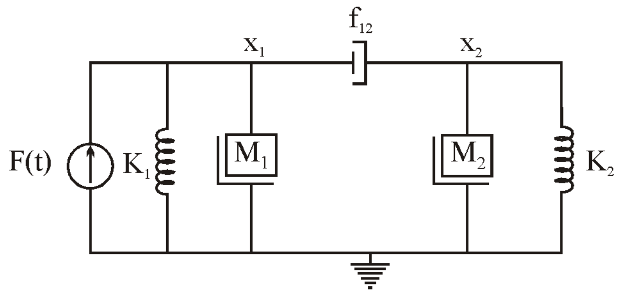
$$\Rightarrow \quad M_2\ddot{x}_2 - f_{12}(\dot{x}_1 - \dot{x}_2) + K_2x_2 = 0 \qquad \text{...(ii)}$$

(a) Differential equations of the mechanical system are

$$M_1\ddot{x}_1 + f_{12}(\dot{x}_1 - \dot{x}_2) + K_1x_1 = F(t)$$

and $$M_2\ddot{x}_2 - f_{12}(\dot{x}_1 - \dot{x}_2) + K_2x_2 = 0$$

(b) Mechanical equivalent representation is shown below




(c) Standard force equation

$$M\ddot{x} + f\dot{x} + Kx = F \quad \text{...(iii)}$$

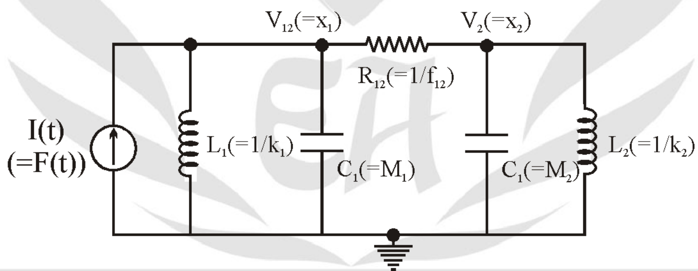
Standard current equation

$$C\ddot{\phi} + \frac{1}{R}\dot{\phi} + \frac{\phi}{L} = I \quad \text{...(iv)}$$

Comparing the above equations (iii) and (iv) we get.

$$M \rightarrow C$$
$$f \rightarrow 1/R$$
$$K \rightarrow 1/L$$
$$F \rightarrow I$$

So, the electrical analogous circuit based on the force-current analogy is given below



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