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ENGINEERS ACADEMY

RPSC - A.En.

Assistant Engineering

CIVIL

Rajasthan Public Service Commission (RPSC)

Volume - 7

Theory of Structure (TOS)

SIERRA INNOVATIONS

DETERMINACY INDETERMINACY

THEORY

1.1 Equation of Static Equilibrium

In a 2-D structure or plane structure (in which all members and forces are in one plane only), the equation of equilibrium are

$$\left.\begin{aligned}\Sigma F_x &= 0\\ \Sigma F_y &= 0\\ \Sigma M_z &= 0\end{aligned}\right\} \text{3 number}$$

In a 3-D structure or space structure (in which members and forces are in 3-D), the equations of equilibrium are

$$\left.\begin{aligned}\Sigma F_x = 0 \quad & \Sigma M_x = 0\\ \Sigma F_y = 0 \quad & \Sigma M_y = 0\\ \Sigma F_z = 0 \quad & \Sigma M_z = 0\end{aligned}\right\} \text{6 number}$$

If member forces cannot be found by equations of static equilibrium alone, the structure is called statically indeterminate.

In this case additional equation needed are obtained by relating the applied loads and reactions to the displacement or slopes known at different points on the structure. These equations are called **compatibility equations.**

1.2 Degree of Static Indeterminacy (D_S)

$$D_S = \begin{pmatrix}\text{No. of unknown forces in members}\\ \text{+ unknown support reactions}\end{pmatrix} - \begin{pmatrix}\text{Available equations of}\\ \text{static equilibrium}\end{pmatrix}$$

and $D_S = D_{Si} + D_{Se}$

Where, D_S = Total indeterminacy

D_{Si} = Degree of internal static indeterminacy

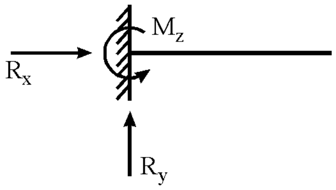
D_{Se} = Degree of external static indeterminacy

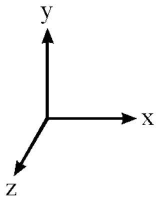
1.3 Support Reactions

Restraining of deformation at support gives rise to support reactions.

1.3.1 Plane Structure

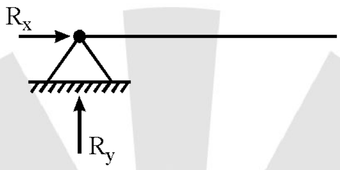
(1) Fixed support





Fixed support restrains Δ_x, Δ_y and θ_{xy}. Hence support reactions are R_x, R_y and M_z (3 nos.)

(2) Pin support or hinged support

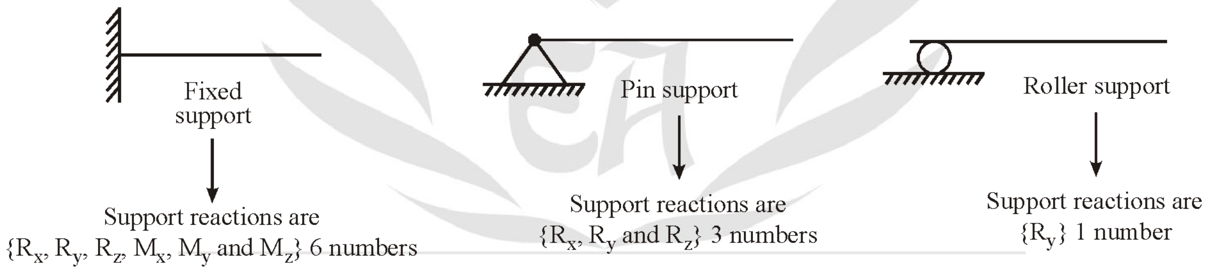


Pin support restrains Δ_x and Δ_y. Hence support reactions are R_x and R_y (2 nos.)

(3) Roller support

Roller support restrains Δ_y. Hence support reaction is R_y.

1.3.2 Space Structure



1.4 External Indeterminacy (D_{Se})

$$D_{Se} = \begin{pmatrix}\text{Total no. of support reactions} \\ \text{in the structure}\end{pmatrix} - \begin{pmatrix}\text{Available equations of} \\ \text{static equilibrium}\end{pmatrix}$$

For plane structure $D_{Se} = R_e - 3$

For space structure $D_{Se} = R_e - 6$

1.5 Internal Indeterminacy (D_{Si})

$$D_{Si} = D_S - D_{Se}$$

= Total indeterminacy – External indeterminacy

1.6 Degree of Static Indeterminacy for Frames

Frames are rigid jointed structures. All the joints are made rigid by providing extra restraint R′. The structure is then cut to make it, *Open Tree* like determinate structure.

For plane frames $D_S = 3C - R'$

For space frames $D_S = 6C - R'$

Where, C = Number of cuts to make structure determinate

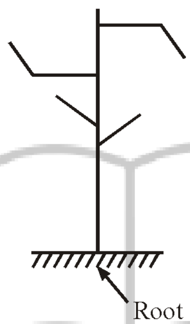
R′ = Number of restraints applied to make all joints rigid.

1.6.1 Open Tree Like Structure

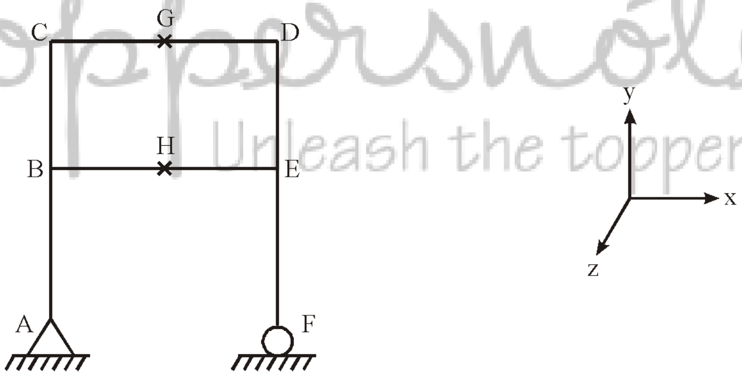
The structure is cut in such a way that each individual cut part looks like a tree as shown below.

Note that :

1. Tree should have only one root.
2. Tree cannot have a closed looped branch.



Ex.:



Since it is a plane frame.

$$D_S = 3 \times 2 - (3) = 3$$

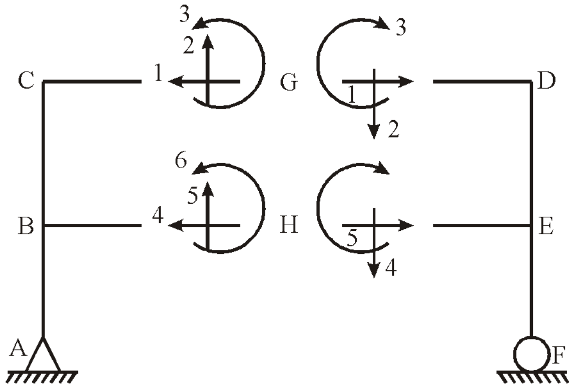
Number of restraint that is required to be applied to make the structure rigid is

1. One number at support A i.e. moment M_{Az}.
2. Two numbers at support F i.e. R_{Fx} and M_{Fz}.

Thus, total number of added restraints = R′ = 1 + 2 = 3

This R′ = 3 corresponds to 3 known reaction conditions i.e. $M_{Az} = 0$, $M_{Fz} = 0$ and $R_{Fx} = 0$.

When the structure is cut it gets divided into two parts.



Open tree like structure

If these 6 reactions at the cut section are known, the structure become completely determinate i.e. forces in all members AB, BC, DE, EF, CD and BE can be determined.

Thus, Number of unknowns = six reactions at the cut section – three known conditions

Where known conditions are $\Sigma R_{Fx} = 0$

$\Sigma M_{Az} = 0$

$\Sigma M_{Fz} = 0$

As the number of cuts are C = 2

Number of unknowns = $3 \times C - R'$

$= 3 \times 2 - 3 = 3$

In 3-D frame, at any cut section number of reactions = 6 [R_x, R_y, R_z, M_x, M_y and Mz]

Hence, Degree of static indeterminacy for 3D frame = $6C - R'$

1.6.2 Restraining Support

For plane and space frames

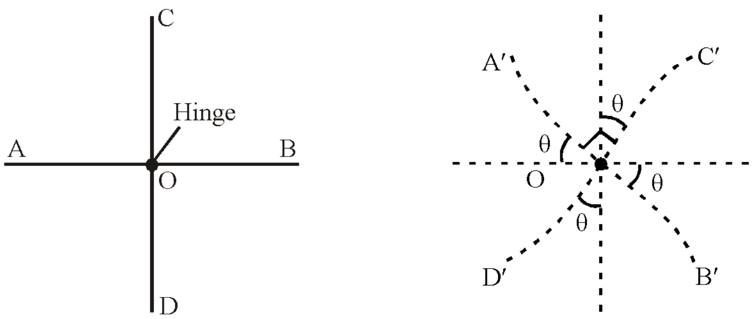
$$\begin{matrix}\text{Number of restraint required}\\ \text{to make a joint rigid}\end{matrix} = \begin{pmatrix}\text{Number of support reactions}\\ \text{for fixed support}\end{pmatrix} - \begin{pmatrix}\text{Number of support reactions}\\ \text{of actual support}\end{pmatrix}$$

1.6.3 Restraining Member/Joint

(1) Plane Frame (Joint having hinge) : Number of restraining moments required at a joint where m-members meet = (m – 1) or no of additional equations gained.

(2) Space Frame (Joints having hinge) : Number of restraining moments required at a joint where m-members meet = 3(m – 1) or no of additional equations gained.

(3) Explanation for Applying (m – 1) Restraining Moment at Joint Having Hinge in 2D Frame : If the joint O had been rigid, rotation of one member with respect to other will be zero as shown in figure below.



However, with joint having hinge, OC, OB and OD will have rotation with respect to OA. To make these three relative rotations zero, we need to apply 3-moments. Thus for 4-members meeting at a joint, number of restraining moments required = 3 = (4 – 1).

Hence, for m-members meeting at a joint number of restraining moments required = (m – 1).

On similar line, it can be shown that in space frame, each member has 3-rotations possible in 3-different planes. Hence with respect to one member number of rotations possible are

$$3m - 3 = 3(m - 1)$$

To restrain these we need to apply 3(m – 1) moments. Hence number of restraining moments required at joint with hinge in 3D-frame = 3(m – 1).

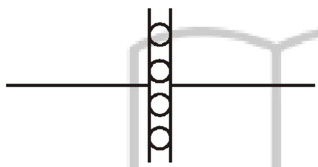
Ex. 1 :

No. of restraint required to be added = 2. They are R_H and M. R_H restrains relative movement.

Ex. 2 :

Number of restraint required to be added = 1 (R_H)

Ex. 3 :



Number of restraint required to be added = 1 (R_V)

1.6.4 Second Method (For Rigid Frame)

In plane frame, every member carries three forces i.e. BM, SF, Axial Force.

Hence Total number of unknowns = $3m + R_e$

Where, m = number of members

and R_e = number of support reactions

At each joint, number of equations of equilibrium available = 3

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_z = 0$$

Total number of equations of equilibrium = 3j

Where, j = number of joints

Hence, degree of static indeterminacy

$$D_S = R_e + 3m - 3j$$

However if the frame carries hinges, then D_S is reduced further by $\Sigma(m' - 1)$, where m′ = number of members meeting at the hinge.

$$D_S = R_e + 3m - 3j - \Sigma(m' - 1)$$

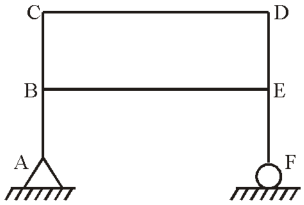
Similarly for space frame $D_S = R_e + 6m - 6j$

Due to presence of internal hinge in space frame

$$D_S = R_e + 6m - 6j - 3\Sigma(m' - 1)$$

Explanation :

(1) Frame without hinge : Let us take a frame as shown below



$$m = \text{number of members} = 6$$

Members are AB, BC, CD, DE, EF and BE.

$$R_e = \text{Number of support reactions} = 3$$

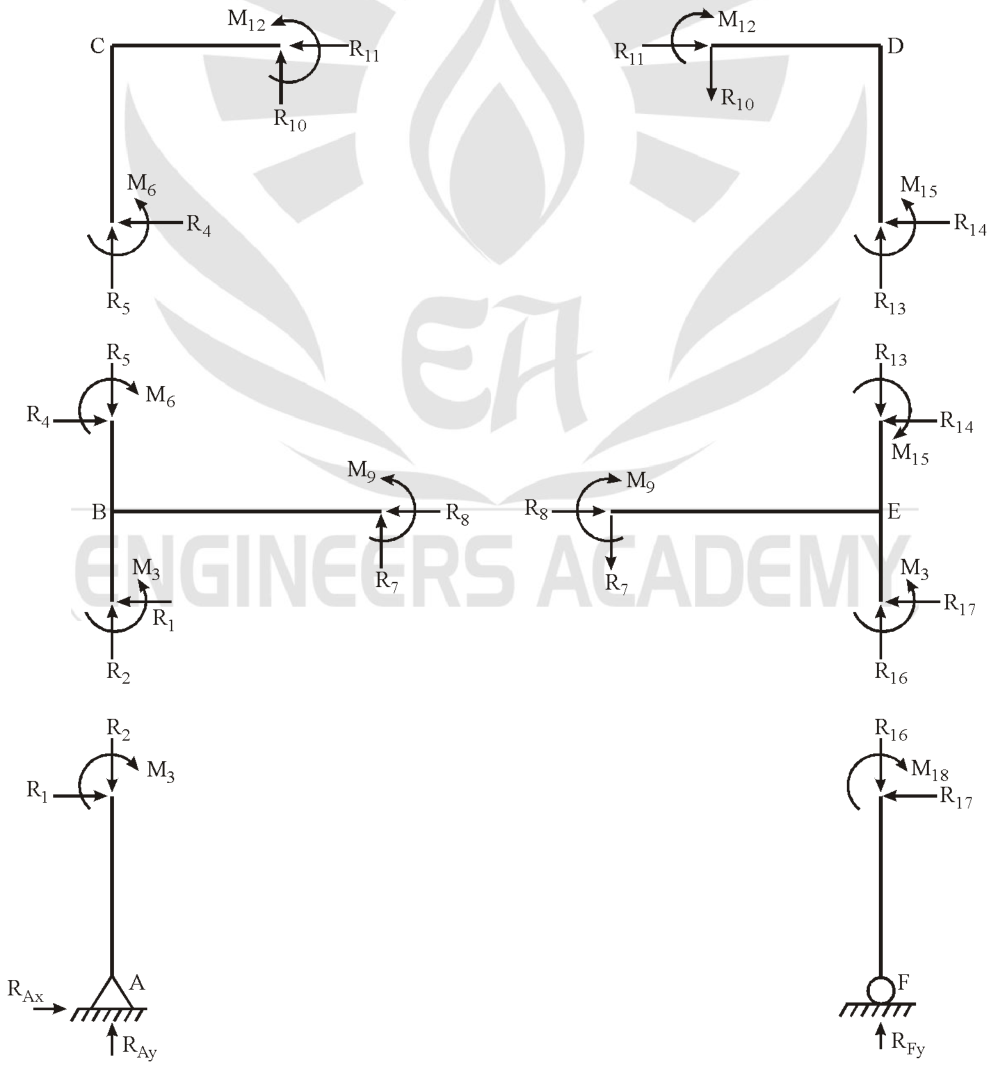
Reactions are R_{Ax}, R_{Ay} and R_{Fy}.

$$j = \text{Number of joints} = 6$$

Joints are A, B, C, D, E and F.

$\therefore$
$$D_S = R_e + 3m - 3j$$
$$= 3 + 3 \times 6 - 3 \times 6 = 3$$

If we make free body diagram of the structure, it looks as below



Note : For each joints one free body diagram has been made.

The structure has been divided into 6-parts which corresponds to 6 number of joints. Each of the six components of free body diagram has to be in equilibrium. Hence for each components three equations of equilibrium are available.

Total number of equations of equilibrium available = 3j = 3 × 6 = 18

Each member of a plane frame carries 3 forces.

Total number of unknown member forces = 3m = 3 × 6 = 18, where m = number of members.

Unknown member forces are R_1, R_2, M_3, R_4, R_5, M_6, R_7, R_8, M_9, R_{10}, R_{11}, M_{12}, R_{13}, R_{14}, M_{15}, R_{16}, R_{17} and M_{18}.

Total number of unknown support reactions $R_e = 3$ i.e. R_{Ax}, R_{Ay} and R_{Fy}.

Hence if all the components of free body diagram are considered then total number of unknowns

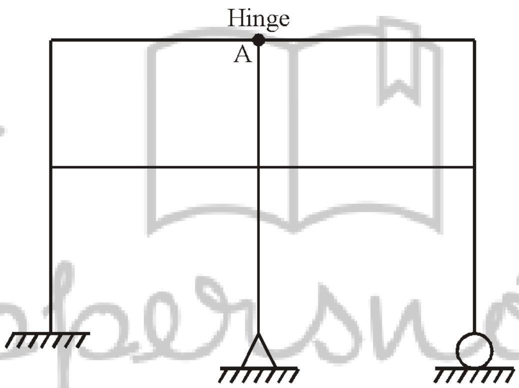
$$= R_e + 3m$$

Total number of known equations = 3j

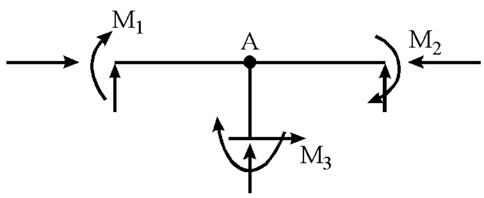
Degree of static indeterminacy

$$D_S = R_e + 3m - 3j$$

(2) Frame with hinges



Considering the above example, had the joint A been rigid, the degree of static indeterminacy would have been $R_e + 3m - 3j$. Due to the presence of joint with hinge, additional independent conditions available are as shown below



Equilibrium of moment at joint A i.e.

$$\Sigma M_A = 0$$

$$M_1 + M_2 + M_3 = 0 \quad \text{...(1)}$$

But it is known that due to presence of hinge at A,

$$M_1 = 0 \quad \text{...(2)}$$

$$M_2 = 0 \quad \text{...(3)}$$

$$M_3 = 0 \quad \text{...(4)}$$

Out of above three conditions (2), (3) and (4), only two are independent because third can be derived from equation (1). Hence number of independent additional equations are two i.e. (m′ – 1), where m′ = number of members meeting at hinge.

Thus, $$D_S = R_e + 3m - 3j - (m' - 1)$$

However if there are more than one hinge, we can have additional independent conditions as $\Sigma(m' - 1)$.

Thus, $$D_S = R_e + 3m - 3j - \Sigma(m' - 1)$$

Note that the hinges that we are considering are not the support hinges. They are member or joint hinges. On similar lines, it can be shown that for 3D-frame.

$$D_S = R_e + 6m - 6j \quad \text{[Frame without hinge]}$$

$$D_S = R_e + 6m - 6j - 3\Sigma(m' - 1) \quad \text{[Frame with hinges]}$$

1.7 Static Indeterminacy for Beams

Beam is made cantilever by adding constraint and removing all other support reactions.

$$D_S = \text{Support removed} - \text{Constraint added}$$
$$= R_e - 3$$

Where 3 is number of equation of equilibrium

and R_e = number of support reactions.

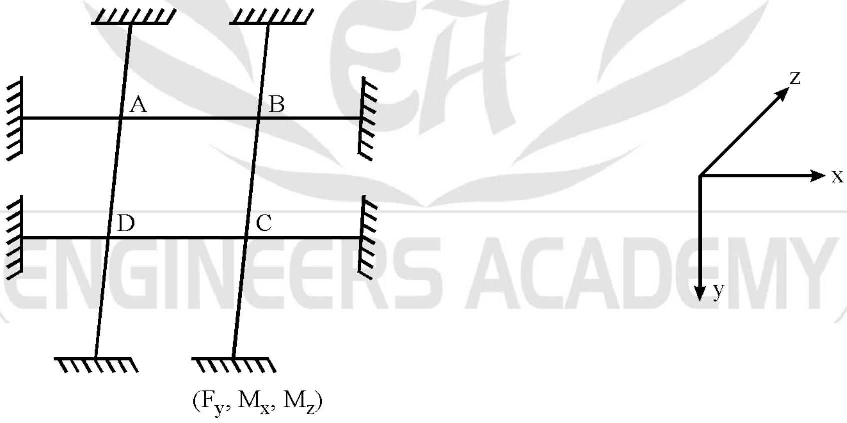
Degree of internal indeterminacy for beams is always zero. Hence

$$D_S = D_{Se} + D_{Si}$$

But $$D_{Si} = 0$$

$\therefore$ $$D_S = D_{Se}$$

1.8 D_S of Horizontal Grid Members Subjected to Vertical Loads



If the members of the horizontal grid are assumed to be rigidly connected and to be subjected to vertical loading only then all the support reaction except M_y, F_x and F_z will vanish. Hence in the members and the supports only M_z, M_x and F_y will remain.

Thus external unknown support reactions are = 8 × 3 = 24 [∵ 8 Number of supports]

Even by knowing F_y, M_x and M_z at all the supports, forces in members AB, BC, CD and DA can not be found. If we cut one of these members, unknowns that will appear will be F_x, M_x and M_z in a member and member force can be found. Hence internal unknowns will be three in number.

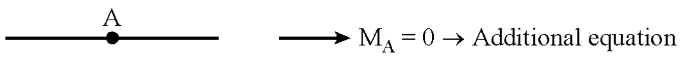
Meaningful number of equations of static equilibrium will be three in number i.e.

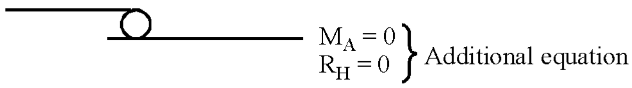
$$\Sigma F_y = 0$$
$$\Sigma M_z = 0$$
$$\Sigma M_x = 0$$

$\therefore$ Degree of static indeterminacy, $D_S = 24 + 3 - 3 = 21$

Note : Another approach of calculation of D_S.

$$D_S = \begin{pmatrix} \text{Number of} \\ \text{external reactions} \end{pmatrix} - \begin{pmatrix} \text{Number of equations} \\ \text{of statical equilibrium} \end{pmatrix} - \begin{pmatrix} \text{Number of} \\ \text{additional equations} \end{pmatrix}$$





1.9 Stability of Structure

Stability of structure is characterized into two parts

- External Stability
- Internal Stability

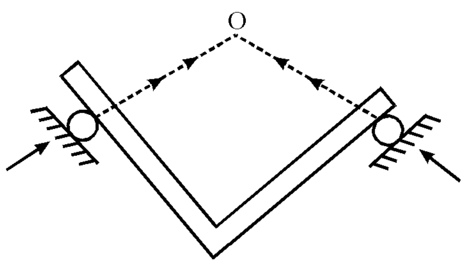
1.9.1 External Stability

If a body is sufficiently constrained by external reactions such that **rigid body movement** of structure can not be occurred, then the structure is said to be externally stable.

Necessary conditions for external stability are

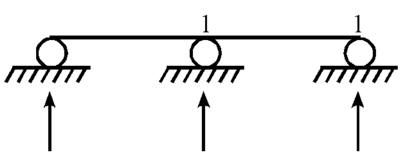
1. There should be minimum three external reactions.
2. Reactions should be
   (i) non-parallel and non-concurrent for plane structure
   (ii) non-parallel, non-concurrent and non-coplanar for space structure.
   (Concurrent means meeting at a single point).

Ex.:



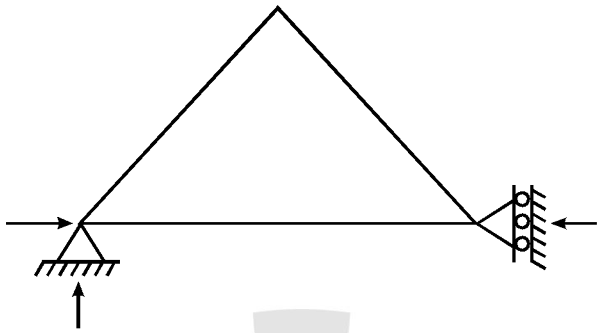
Reaction are concurrent, hence structure is externally unstable.

Ex.:



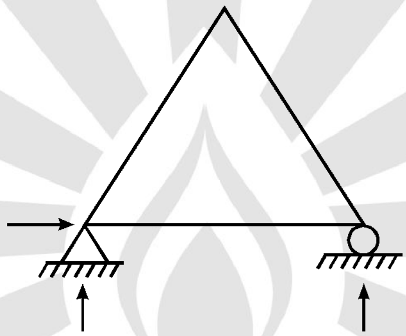
Reaction are parallel, hence inclined loading will lead to rigid body movement. So structure is externally unstable.

Ex.:



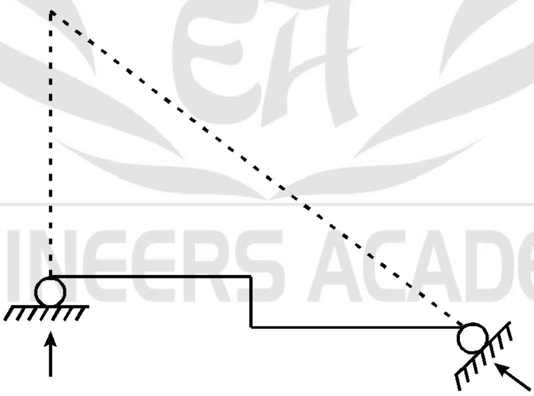
Reactions are concurrent, hence structure is externally unstable.

Ex.:



Three reactions are present that are non parallel and non-concurrent. Hence structure is externally stable.

Ex.:

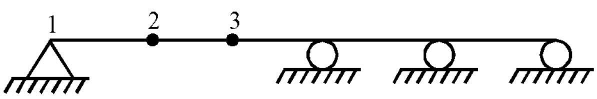


Reactions are concurrent, hence structure is externally unstable.

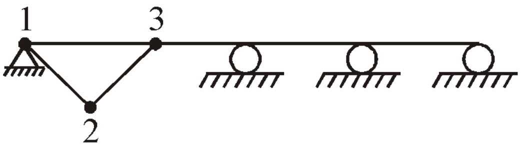
1.9.2 Internal Stability

When a part of the structure moves appreciably with respect to the other part, then the structure is said to be internally unstable.

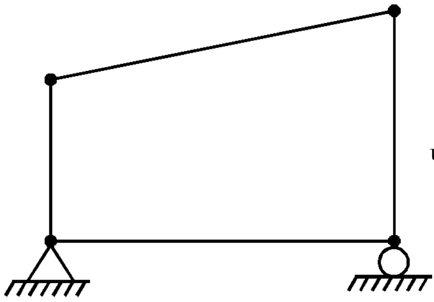
Ex.:



Three consecutive hinges are present that is a mechanism. The failure condition is shown below

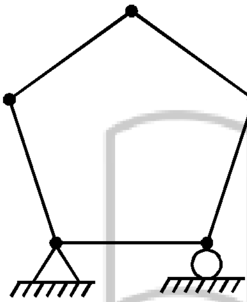


Ex.:



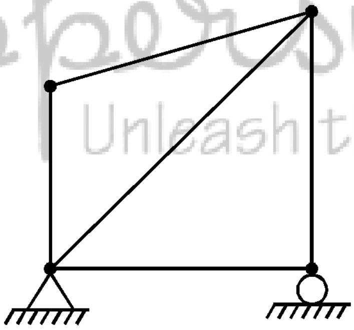
unstable

Ex.:



unstable

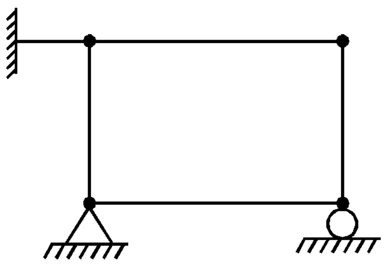
Ex.:



stable

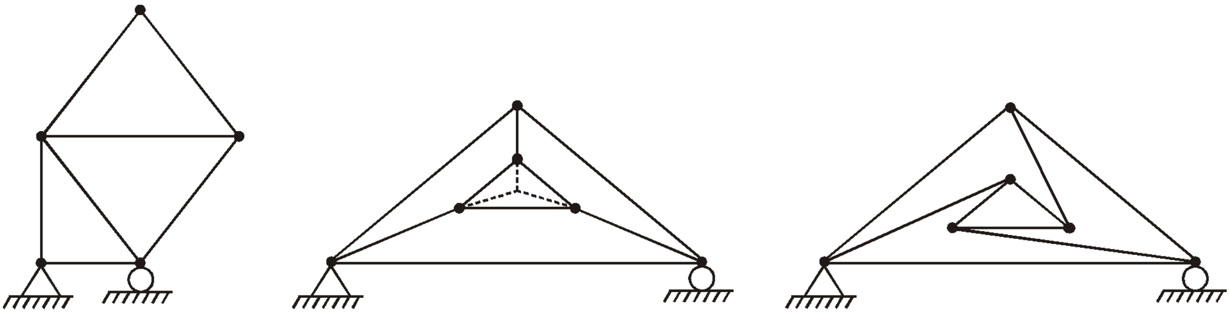
Note that application of a bracing has ensured that one part does not move appreciably with respect to the other part.

Ex.:



stable

Fixing of one joint has ensured that appreciable deformation of one part cannot take place with respect to other.



stable unstable unstable

Note : At zero load condition there should be zero stress in all members but existence of non zero solution violates the principle of unique solution.

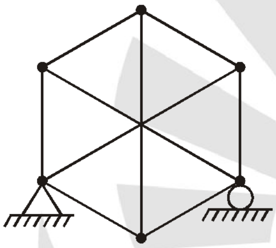


Fig. : (a) Unstable if all diagonals are of equal length

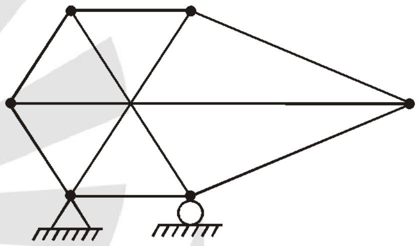
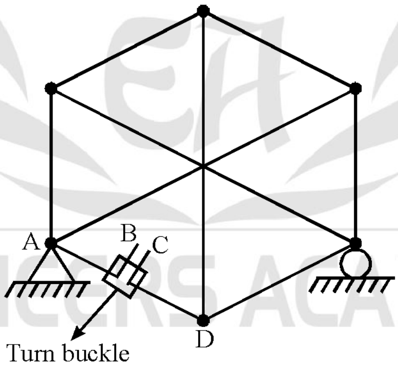


Fig. : (b) Stable

Figure (a) is unstable because if 1 kN load is applied in any member by turn buckle application, equilibrium will be satisfied at all joints without developing any joint reaction.



- Turn buckle is an arrangement which is used to apply tension. By rotating the turn buckle, end B and C will be brought closer, hence tension is created in AD.
- Now if 2 kN load is applied again equation is satisfied at all joints without developing any joint reaction.
- Thus for all value of load, reactions developed are zero. This implies that there is no unique value of reaction obtained for a particular loading. This is the characteristic of unstable structure.

Note : It is not always easy to visualize by visual inspection, whether the structure is unstable or stable. In such cases check is to analyze the structure and if no unique solution is achieved, the structure is unstable.

1.10 Static Indeterminacy of Trusses

A truss is designed in such a way that members of truss always carries only **axial forces.** Hence at truss joints equations of equilibrium available

$$\begin{array}{c}\text{Number of available}\\ \text{equilibrium equation}\end{array} = \begin{cases} 2 & \text{for plane truss i.e. } \quad \Sigma F_x = 0, \Sigma F_y = 0 \\ 3 & \text{for space truss i.e. } \quad \Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0 \end{cases}$$

Number of unknowns in truss is = $R_e + m$

Where, m = Number of members (because each member carry one force)

R_e = Number of reactions at supports.

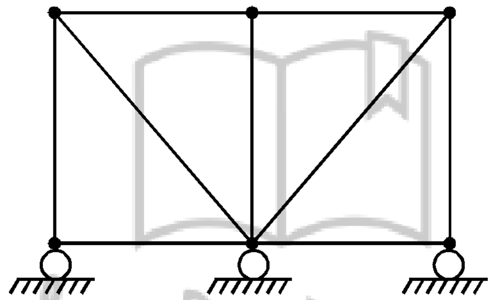
Hence $R_e + m - 2j = 0$ statically determinate plane-truss ...(1)

$R_e + m - 2j > 0$ statically indeterminate plane-truss ...(2)

$R_e + m - 2j < 0$ unstable truss ...(3)

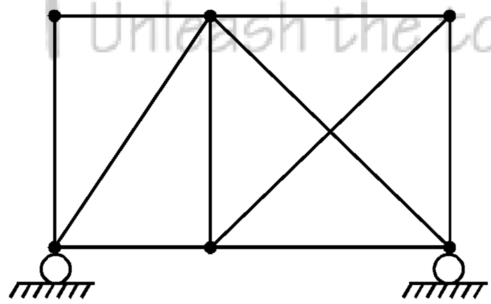
However, condition (1) and (2) does not ensure that the truss will be stable. The stability should be checked visually or analytically.

Ex.:



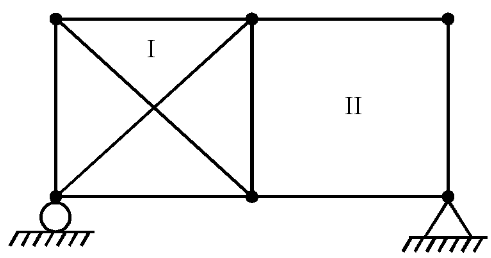
Externally unstable because of parallel support reactions. Hence determinate but unstable.

Ex.:



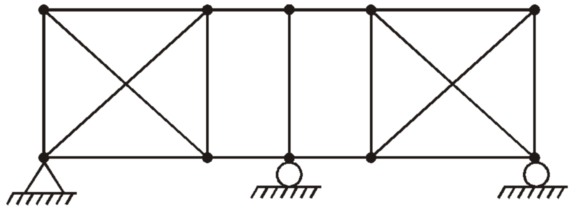
Determinate but externally unstable because of parallel support reactions.

Ex.:



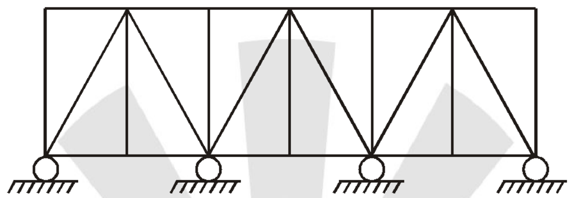
Internally unstable because unbraced portion (II) can have appreciable deformation.

Ex.:



Unstable internally

Ex.:



Externally unstable

1.10.1 Internal and External Indeterminacy in Plane Truss

$$\text{External indeterminacy} = R_e - 3$$

Where, R_e = number of support reactions.

But degree of static indeterminacy

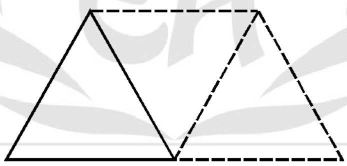
$$D_S = R_e + m - 2j$$

$$= [R_e - 3] + [m - (2j - 3)]$$

Where external indeterminacy $D_{Se} = R_e - 3$

and internal indeterminacy $D_{Si} = m - (2j - 3) = m - 2j + 3$

Note : Basic structure of a plane truss is a triangle.



To this triangle two members and one joint are added to build up the truss further. Hence stable configuration of this **simple truss** is obtained when

$$m = 3 + 2\,(j - 3) = 2j - 3$$

Thus, if $m = (2j - 3)$, truss is internally determinate and stable.

$m > (2j - 3)$, truss is internally indeterminate and stable.

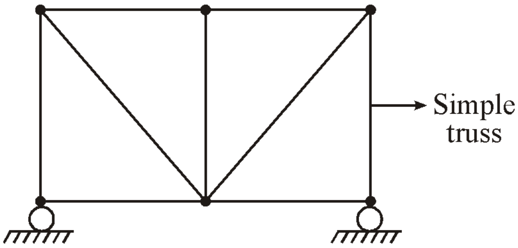
$m < (2j - 3)$, truss is internally unstable.

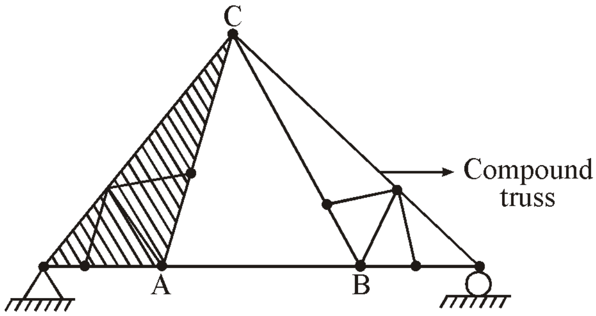
Simple, Compound and Complex truss :

Simple Truss : In a triangle when two bar and one joint are progressively added to form a truss, this truss is called simple truss.

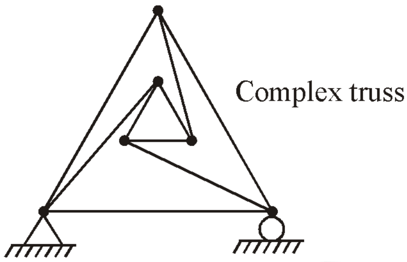
Compound Truss : Two simple truss connected by a set of joints and bars.

Complex Truss : There is no joint where only two bars meet.



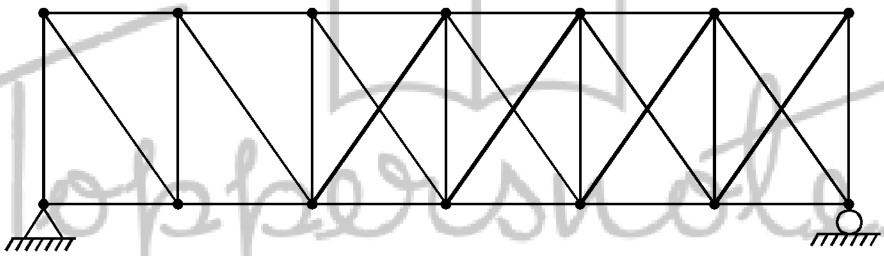


Shaded portion is a simple truss. Two such truss have been joined by using a bar AB and joint C.



Note (i) : In a simple truss [m – (2j – 3)] is number of panels with double diagonals.

Hence, D_S = (External indeterminacy) + (Number panels with double diagonal)



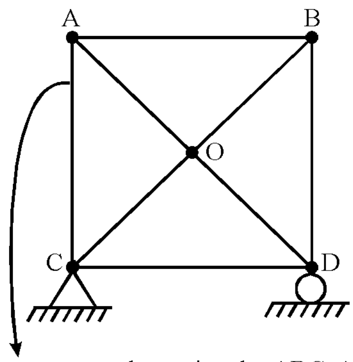
D_S = External indeterminacy + Number of panels of double diagonal = 0 + 4 = 4

Also, $m = 29,\ R_e = 3,\ j = 14$

$$D_S = R_e + m - 2j$$

$$= 3 + 29 - 2 \times 14 = 4$$

Note (ii) : Truss having members which cross-over each other or members that serves as side for more than two triangles are likely to be indeterminate.



Side AC is common to three triangles ABC, ACD, ACO

1.10.2 Internal and External Indeterminacy in Space Truss

$$D_S = R_e + m - 3j$$
$$= [R_e - 6] + [m - (3j - 6)]$$

Where external indeterminacy $D_{Se} = R_e - 6$

and internal indeterminacy $D_{Si} = m - (3j - 6) = m - 3j + 6$

If $D_S = 0$ Truss is stable and determinate but stability should be checked visually

$D_S > 0$ Truss is indeterminate

$D_S < 0$ Truss is unstable

1.11 Kinematic Indeterminacy

In stiffness method of analysis, joint displacements are taken as basic unknowns. Load displacement equations (i.e. relation between load applied and displacement of joint) are written and by using equilibrium equations these joints displacement and hence member forces are found.

Thus number of unknown joint displacement are required to be known to proceed with the analysis. Number of these unknown joint displacement are called unknown degree of freedom or degree of kinematic indeterminacy (D_k)

$$D_k = \begin{pmatrix}\text{Total possible degrees} \\ \text{of freedom}\end{pmatrix} - \begin{pmatrix}\text{Number of available support reactions which are} \\ \text{generated as the result of restrained joint displacements}\end{pmatrix}$$

Types of joints	Total possible degree of freedom
• Rigid jointed plane frame	Δ_x, Δ_y and θ = 3 nos.
• Rigid jointed space frame	Δ_x, Δ_y, Δ_z, θ_x, θ_y and θ_z = 6 nos.
• Pin jointed plane structure	Δ_x and Δ_y = 2 nos.
• Pin jointed space structure	Δ_x, Δ_y and Δ_z = 3 nos.

$D_k = 2j - R_e$ Pin jointed plane frame

$= 3j - R_e$ Pin jointed space frame

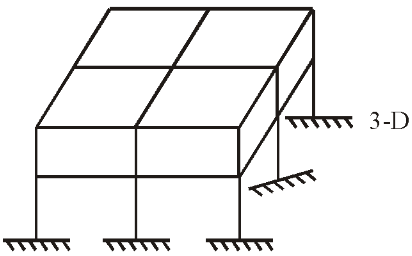
$= 3j - R_e$ Rigid jointed plane frame

$= 6j - R_e$ Rigid jointed space frame

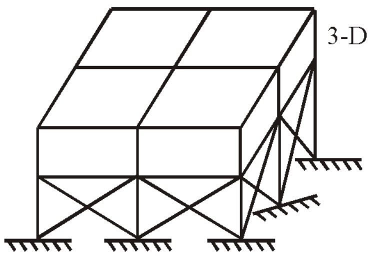
Where j = Number of joints

R_e = Number of support reactions

Note (i) : Braced and unbraced frames

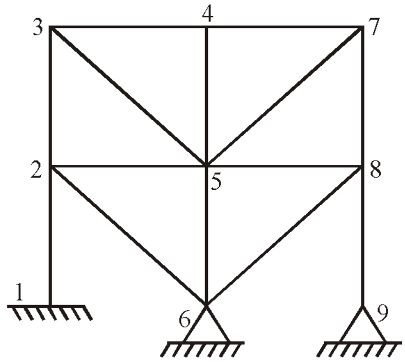


Unbraced frame



Braced frame

Note (ii) : If non extensibility of member is adopted, the joints of triangulated rigid jointed frame cannot have linear displacements.



Because linear displacements of joints are restrained by use of braces, possible joints displacements are θ_2, θ_3, θ_4, θ_5, θ_6, θ_7, θ_8 and θ_9.

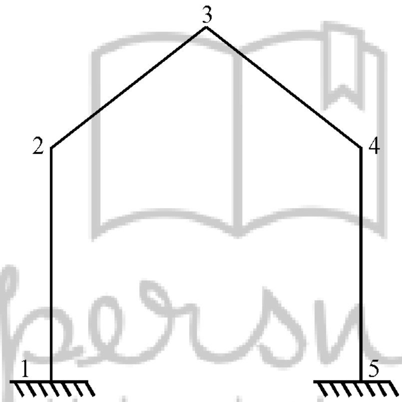
$\therefore$ $$D_k = 8$$

1.11.1 For Space Frame (Unbraced Building Frame)

$$D_k = 6j - R_e - m'$$

Where, m' = number of non extensible members.

For gabled frame



If non extensibility of member is considered then

$\Delta y_2 = 0$ from non extensibility of member 1-2

$\Delta y_4 = 0$ from non extensibility of member 4-5

Out of Δx_2, Δx_3, Δx_4 and Δy_3 only two are independent because from the non extensibility of members 2-3 and 3-4 two equations can be derived relating these four displacements.

Hence $$D_k = 9 - (2 + 2) = 5$$

This relationship could also have been obtained simply by using the formula

$$D_k = 3j - R_e - m'$$
$$= 3 \times 5 - 6 - 4 = 5$$

Note : Other method of finding degree of kinematic indeterminacy when all members are inextensible is

$$D_k = \text{Number of possible joint rotation} + \text{Number of sway}$$

Hence in the gable frame as shown above

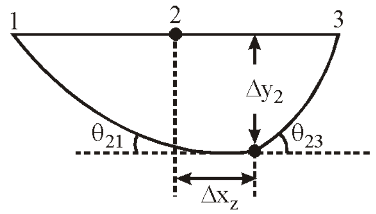
Number of possible joint rotation = 3 i.e. θ_2, θ_3 and θ_4

Number of possible sway = 2 i.e. (sway of joints 2 and 4) and (sway of joint 3)

$\Rightarrow$ $$D_k = 3 + 2 = 5$$

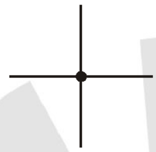
1.11.2 Rigid Jointed Frames with Internal Member Hinge or Joint Hinge

In this case additional degrees of freedom are created. Each internal member hinge will add 4-degree of freedom in **plane frame**.



They are θ_{21}, θ_{23}, Δy_2, Δx_2

Each joint hinge will add (m′ − 1) additional degree of freedom in plane frame.



$m' - 1 = 4 - 1 = 3$

Where, m′ = number of member meeting at hinge.

Each internal hinge will add 9-DoF in space frame. They are Δx, Δy, Δz and six rotations.

Each joint hinge will add 3(m′ − 1) additional DoF in member in space frame.

Hence, For 2-D $D_k = (3j - R_e)$ + additional DoF

For 3-D $D_k = (6j - R_e)$ + additional DoF

If non extensibility is also considered then

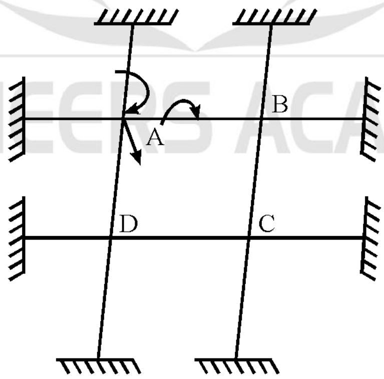
Hence, For 2-D $D_k = (3j - R_e)$ + additional DoF − m″

For 3-D $D_k = (6j - R_e)$ + additional DoF − m″

Where, m″ = number of inextensible members

1.11.3 D_k for Horizontal Grid Subjected to Vertical Loads Only

If a rigid-jointed grid is subjected to loads in perpendicular direction to the plane of the grid only, each joint can have three displacements components. They are translation perpendicular to the plane of the grid and rotation about two orthogonal axes in the plane of grid.



grid in horizontal plane

Fig. : Load applied is perpendicular to the plane of the grid.

The above grid is kinematically indeterminate to 12th degree.