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1st – Grade

Mathematics

School Education

Rajasthan Public Service Commission

Paper – 2

Volume - 3

SIERRA INNOVATIONS

विषय सूची

क्र.सं.	अध्याय	पृष्ठ सं.
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सदिश कलन

यदि सदिश $\vec{r}$, अदिश t का फलन हो।

Q.) यदि $\vec{r} = 3t^2\hat{i} + (t^3+1)\hat{j} + 4t\hat{k}$ हो then

i) $\frac{d\vec{r}}{dt} = 6t\hat{i} + 3t^2\hat{j} + 4\hat{k}$

ii) $\left|\frac{d\vec{r}}{dt}\right|_{t=1 \text{ पर}} = 6\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \left|\frac{d\vec{r}}{dt}\right|_{t=1} = \sqrt{36+9+16} = \sqrt{61}$

iii) $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$

$= 6\hat{i} + 6t\hat{j} + 0\hat{k}$

$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds}$

वक्र $\vec{r} = f(t)$ के किसी बिन्दु पर अवकलज $\frac{d\vec{r}}{dt}$ की दिशा, $d\vec{s}$ के इस बिन्दु पर स्पर्शी की दिशा में होती है।

$\therefore$ वक्र $\vec{r} = f(t)$ के point $P(t)$ पर स्पर्शी की दिशा में इकाई सदिश $\frac{\left(\frac{d\vec{r}}{dt}\right)}{\left(\frac{d\vec{r}}{dt}\right)}$

Q.) यदि $\vec{r} = (x^2yz)\hat{i} + (xy^2z)\hat{j} + (xyz^2)\hat{k}$ हो

then i) $\frac{d\vec{r}}{dx} = (2xyz)\hat{i} + (y^2z)\hat{j} + (yz^2)\hat{k}$

ii) $\frac{d\vec{r}}{dz} = (x^2y)\hat{i} + (xy^2)\hat{j} + (2xy)\hat{k}$

(iii) $\frac{\partial^2 \vec{r}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left[(x^2z)\hat{i} + (2xyz)\hat{j} + (xz^2)\hat{k}\right]$

$= (2xz)\hat{i} + (2yz)\hat{j} + x(z^2)\hat{k}$

(iv) $\frac{\partial^2 \vec{r}}{\partial x^2} = (2yz)\hat{i} + 0 + 0$

(v) $\frac{\partial^2 \vec{r}}{\partial y^2} = (2xz)\hat{j}$

Q.42) $\vec{r} = (\cos nt)\hat{i} + (\sin nt)\hat{j}$

$\vec{r} \times \frac{d\vec{r}}{dt} = \left[(\cos nt)\hat{i} + \sin nt\,\hat{j}\right] \times \left[-n\sin nt\,\hat{i} + n\cos nt\,\hat{j}\right]$

$\because \hat{i}\times\hat{i} = 0$, $\hat{j}\times\hat{i} = 0$

$= n\cos^2 nt\,\hat{k} + n\sin^2 nt\,\hat{k}$

$= n(1)\hat{k} = n\hat{k}$

Result —

① $\frac{d}{dt}(\vec{a})^2 = 2\vec{a}\frac{d\vec{a}}{dt}$

prove $\because (\vec{a})^2 = \vec{a}\cdot\vec{a}$

L.H.S. $\frac{d}{dt}(\vec{a}\cdot\vec{a}) = \vec{a}\cdot\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{a}$

$= \vec{a}\cdot\frac{d\vec{a}}{dt} + \vec{a}\frac{d\vec{a}}{dt}$

$= 2\vec{a}\frac{d\vec{a}}{dt}$

② $\vec{a}\cdot\frac{d\vec{a}}{dt} = a\left(\frac{da}{dt}\right)$ — ①

prove $\because (\vec{a})^2 = \vec{a}\cdot\vec{a} = a\,a\cos 0 = a^2$

$\therefore \frac{d}{dt}(\vec{a})^2 = \frac{d}{dt}(a^2) = 2a\frac{da}{dt}$ — ②

by ① & ② $\Rightarrow \boxed{\vec{a}\cdot\frac{d\vec{a}}{dt} = a\left(\frac{da}{dt}\right)}$

Q] ~~यदि फलन~~ $f(x,y,z) = x^2y^3z^4$ ~~एक अदिश फलन है~~ then ∇f ज्ञात करो।

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^3z^4)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y^3z^4) + \hat{j}\frac{\partial}{\partial y}(x^2y^3z^4) + \hat{k}\frac{\partial}{\partial z}(x^2y^3z^4)$$

$$\boxed{\nabla f = 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}}$$

Q] यदि $f(x,y,z) = x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}$ है then (i) ∇f व (ii) $\nabla \times f$ ज्ञात करो।

Soln (i) $\nabla \cdot f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k})$

$\hat{i}\cdot\hat{i} = 1$

$$= 2xyz + 2xyz + 2xyz$$

$$= 6xyz$$

(ii) $\nabla \times f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k})$

$\hat{i}\times\hat{j} = \hat{k}$

$\hat{i}\times\hat{i} = 0$

$$= y^2z\hat{k} + yz^2\hat{j} + (-x^2z\hat{k}) -$$

$$\nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \hat{i}(xz^2 - xy^2) - \hat{j}(yz^2 - x^2y) + \hat{k}(y^2z - x^2z)$$

$$\nabla \times f = x(z^2 - y^2)\hat{i} - y(z^2 - x^2)\hat{j} + z(y^2 - x^2)\hat{k}$$

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प्रवणता पर प्रमेय ⇒

(V) अदिश फलन $\phi(x,y,z)$ की प्रवणता $\nabla\phi$ एक सदिश है।

Prove Let एक अदिश फलन $\phi(x,y,z)=c$ है।

$$\therefore d\phi(x)=0$$

$$\Rightarrow \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\phi\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)=0$$

$$\Rightarrow (\nabla\phi)\cdot(d\vec{r})=0 \qquad (\because \vec{r}=x\hat{i}+y\hat{j}+z\hat{k})$$

$\Rightarrow$ ~~$\nabla\phi$~~ $\nabla\phi$, सदिश $d\vec{r}$ के $\perp$ है।

$\Rightarrow \nabla\phi$, सतत $\phi(x,y,z)=c$ के बिन्दु $p(\vec{r})$ पर अभिलम्ब की दिशा में सदिश है।

समतल का समी.

Q.12) Let given Surface $\phi: xyz-4=0$ है।

$$\therefore \nabla\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(xyz-4)$$

$$=yz\hat{i}+xz\hat{j}+xy\hat{k}$$

$\therefore$ point $(1,2,2)$ पर-

$$\nabla\phi=4\hat{i}+2\hat{j}+2\hat{k}$$

$\therefore$ दी सतह के बिन्दु $(1,2,2)$ पर स्पर्श तल का समी. $\Rightarrow$ $4(x-1)+2(y-2)+2(z-2)$

$$\Rightarrow \boxed{4x+2y+2z=12}$$

Q.13) point $P(1,2,2)$ पर अभिलम्ब का समी.

$$\frac{x-1}{4}=\frac{y-2}{2}=\frac{z-2}{2}$$

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या $\boxed{\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}}$

प्रवणता पर आधारित परिणाम –

① $\text{grad}\, r = \nabla r = \hat{r}$

proof $\nabla r = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(r)$

$= \hat{i}\frac{dr}{dx} + \hat{j}\frac{dr}{dy} + \hat{k}\frac{dr}{dz}$

$= \left(\frac{x}{r}\right)\hat{i} + \left(\frac{y}{r}\right)\hat{j} + \left(\frac{z}{r}\right)\hat{k}$

$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r} = \hat{r}$

② $\nabla f(r) = \text{grade}\cdot f(r) = f'(r)\nabla r$

$\nabla f(r) = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right) f(r)$

$= \hat{i} f'(r)\frac{dr}{dx} + \hat{j} f'(r)\frac{dr}{dy} + \hat{k} f'(r)\frac{dr}{dz}$

$= f'(r)\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$

$= f'(r)\,\hat{r}$

या $\boxed{\nabla f(r) = f'(r)\nabla r}$

Special case – जब $f(r) = \frac{1}{r}$ हो तब

① $\nabla\left(\frac{1}{r}\right) = \left(-\frac{1}{r^2}\right)\cdot\hat{r}$

$= \left(-\frac{1}{r^3}\right) r\,\hat{r}$

$\boxed{\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^2}}$

② $\boxed{\nabla(e^r) = e^r(\hat{r})}$

③ $\boxed{\nabla(\log r) = \frac{1}{r}\cdot\hat{r}}$

(4) $\nabla(r^m) = m r^{m-1} \cdot \hat{r}$

$= m r^{m-1} \nabla r$

Q.47)

$\nabla e^{r^2} = e^{r^2} \cdot \hat{r}$

जाग ले → $= 2r \cdot e^{r^2} \cdot \hat{r}$

$= 2 e^{r^2} \cdot r \cdot \frac{\vec{r}}{r}$

$= 2 e^{r^2} \cdot \vec{r}$

Q.1) $\vec{r} = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$

$\left|\frac{d\vec{r}}{dt}\right| = \cos t\,\hat{i} - \sin t\,\hat{j} + \hat{k}$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t + 1}$

$= \sqrt{1+1} = \sqrt{2}$

$\vec{r} = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 6t\hat{k}$

$\vec{v} = \frac{d\vec{r}}{dt} = -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 6\hat{k}$

त्वरण $= \frac{d\vec{v}}{dt} = -4\cos t\,\hat{i} - 4\sin t\,\hat{j}$

$t = 1$ पर →

$\left|\frac{d\vec{v}}{dt}\right|_{t=1} = \sqrt{\quad}$

Q.2) $\vec{r} = 3\hat{i} - 6t^2\hat{j} + 4t\hat{k}$

$\frac{d\vec{r}}{dt} = -12t\,\hat{j} + 4\hat{k}$

$\frac{d^2\vec{r}}{dt} = -12\hat{j} + 0 = -12\hat{j}$

Q.3) किसी सदिश $f(t)$ का परिमाण अचर होगा iff $F \cdot \frac{dF}{dt} = 0$

Q.4) $\vec{a} = 5t^2\hat{i} - \cos t\,\hat{j}$

$\vec{b} = t\hat{i} - \sin t\,\hat{j}$

$(\vec{a} \cdot \vec{b}) = 5t^3 + \sin t \cos t = 5t^3 + \frac{\sin 2t}{2}$

$\frac{d}{dt}(\vec{a} \cdot \vec{b}) = 15t^2 + \frac{2\cos 2t}{2}$

Q.5) $x = 4\cos t$, $y = 4\sin t$, $z = 6t$

वक्र पर किसी पर बिन्दु का स्पर्शी सदिश →

Q.8) $f(x,y,z) = x^3 - y^3 + x^2 z$

$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^3 - y^3 + x^2 z)$

$= 3x^2\hat{i} - 3y^2\hat{j} + x^2\hat{k}$

$(\nabla f)_{(1,-1,2)} = 3\hat{i} - 3\hat{j} + \hat{k}$

Q.9) $f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial z}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

$= \left(\frac{2x}{a^2}\hat{i} + \frac{2y}{b^2}\hat{j} + \frac{2z}{c^2}\hat{k}\right)$

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Q.10) $f = xy + yz + zx$

$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(xy + yz + zx)$

$\nabla f = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

point $(3,1,2)$ पर —

$\nabla f = 3\hat{i} + 5\hat{j} + 4\hat{k}$ — (1)

Now $2\hat{i} + 3\hat{j} + 6\hat{k}$ की दिशा में इकाई सदिश —

$$\hat{a} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + 3^2 + 6^2}}$$

$$\Rightarrow \hat{a} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7} \quad — (2)$$

अतः अभीष्ट दिक् अवकलज

$$\frac{d\phi}{ds} = \text{grade}\,\phi \cdot \hat{a}$$

$$= (5\hat{i} + 5\hat{j} + 4\hat{k}) \cdot \frac{(2\hat{i} + 3\hat{j} + 6\hat{k})}{7}$$

$$= \frac{6 + 15 + 24}{7} = \frac{45}{7}$$

Q.11) $\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2y + 2xy - 4)$

$\nabla f = (2xy + 2x)\hat{i} + (x^2 + 2x)\hat{j}$ — (1)

∵ अभिलम्ब का सदिश समी.

$$(\vec{r} - \vec{r}_0) \times \nabla f = \vec{0} \quad — (2)$$

∵ point $(1,-2,3)$ पर —

$\text{grade}\, f = \{2(1)(-2) + 2(1)\}\hat{i} + (1+2)\hat{j}$

$\nabla f = -2\hat{i} + 6\hat{j}$ — (3)

$\vec{r}_0 = \hat{i} - 2\hat{j} + 3\hat{k}$ — (4)

put in (2) $\Rightarrow$

$\Rightarrow \{(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k})\} \times (-2\hat{i} + 6\hat{j})$

$\Rightarrow \{(x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k}\} \times (-2\hat{i} + 6\hat{j})$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-1 & y+2 & z-3 \\ -2 & 6 & 0 \end{vmatrix} = \vec{0}$$

$\Rightarrow \hat{i}(0 - 6z + 18) - \hat{j}(0 + 2z - 6) + \hat{k}(6x - 6 + 2y + 4)$

$\Rightarrow 6(z+3)\hat{i} - 2(z-3)\hat{j} + 2(3x + y - 1)\hat{k}$

Q.11) ∵ given $\phi = x^2y + 2xz - 4$

$\nabla\phi = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(x^2y + 2xz - 4)$

$= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$

at point $(1,-2,3)$ $\Rightarrow$

$\nabla\phi = 2\hat{i} + \hat{j} + 2\hat{k}$

∴ पतह के point $(1,-2,3)$ पर अभिलम्ब एक सदिश $\Rightarrow$

$$\hat{n} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}}$$

$$\hat{n} = \frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k}) \quad \text{Ans.}$$

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Q.14) $\vec{f} = xi + (x+y)j + (x+y+z)k$

$\text{div} f = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}(x+y) + \frac{\partial}{\partial z}(x+y+z)\right)$

$= \hat{i} + \hat{j} + \hat{k}$ $\quad 1+1+1$

$= 3$

Q.17) $\text{div}\,\vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x\hat{i}+y\hat{j}+z\hat{k})$

$= \left(\frac{d\hat{i}}{\partial x} + \frac{d\hat{j}}{\partial y} + \frac{d\hat{k}}{\partial z}\right)\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$

$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$

$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$

$= \sum \left[1 \cdot \frac{1}{r} + x\left(\frac{-1}{r^2}\right)\frac{x}{r}\right]$

$= \sum \frac{1}{r} - \frac{1}{r^3}\sum x^2$

$= \frac{3}{r} - \frac{1}{r^3}(r^2) = \frac{2}{r}$

Q.18) $f = (xy^2 + yz^3)$

$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$

$\text{grad} f = \hat{i}\,y^2 + (2xy + z^3)\hat{j} + (3yz^2)\hat{k}$

Now point $(2,-1,1)$ पर —

$\text{grade}\,f = \hat{i} + (-4+1)\hat{j} + (-3)\hat{k}$

$\nabla f = \hat{i} - 3\hat{j} - 3\hat{k}$

साथ ही $\hat{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ की दिशा में एक सदिश $\Rightarrow$

$= \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}}$

$= \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

अतः अभिष्ट दिक अवकलज

$\frac{df}{ds} = (\nabla f)\,\hat{a}$

$= (\hat{i} - 3\hat{j} - 3\hat{k})\left(\frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}\right)$

$= \frac{1}{3}(1 - 6 - 6) = -11/3.$

Q.4+) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = |\vec{r}|$ हो तब

$\nabla(e^{r^2}) = \left(i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right)(e^{r^2})$

$= \sum\left[i\frac{d}{dx}(e^{r^2})\right]$

$= \sum i\left(\sum\left(e^{r^2} \cdot 2r\frac{dr}{dx}\right)\right.$

$= 2r \cdot e^{r^2} \cdot \sum\left(i\frac{x}{r}\right)$

$= 2 \cdot e^{r^2}\left(\frac{\vec{r}}{r}\right)$

$= 2 \cdot e^{r^2}(\vec{r})$

$\nabla(e^{r^2}) = e^{r^2}(2r) \cdot (\nabla r)$

$= 2r \cdot e^{r^2}\left(\frac{\vec{r}}{r}\right)$

$= 2e^{r^2} \cdot \vec{r}$

Q.48) $\phi = xy + yz + zx$

$\therefore \nabla\phi = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{j}\frac{d}{dz}\right)$

$(xy + yz + zx)$

$\nabla\phi = \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(x+y)$

$\because$ point $A(1,2,0)$ पर —

$\nabla\phi = 2\hat{i} + \hat{j} + 3\hat{k}$

Now ~~$2\hat{i}+\hat{j}+3\hat{k}$~~ की दिशा $\hat{i} + 2\hat{j} + 2\hat{k}$ में इकाई सदिश

$\vec{a} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{3}$

Now फलन ϕ का $\vec{a}$ की दिशा में दिक अवकलज

$= \nabla\phi \cdot \vec{a}$

$\Rightarrow (2\hat{i} + \hat{j} + 3\hat{k})\left(\frac{\hat{i} + 2\hat{j} + 3\hat{k}}{3}\right) = 10/3$

page-8 यदि $\vec{a}$ व $\vec{b}$ दो अचर सदिश हो & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ हो तब –

(1) $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$

proof – $\nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(a_1x + a_2y + a_3z)$

$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$

$\therefore \boxed{\nabla(\vec{a}\cdot\vec{r}) = \vec{a}}$ या $\boxed{\nabla(\vec{r}\cdot\vec{a}) = \vec{a}}$

(2) $\nabla[\vec{r}\cdot\vec{a}\cdot\vec{b}] = \vec{a}\times\vec{b}$

$\Rightarrow \nabla\{\vec{r}\cdot(\vec{a}\times\vec{b})\} = \vec{a}\times\vec{b}$ {result ii}

(3) $\vec{a}\cdot\nabla = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot\left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)$

$= a_1\frac{d}{dx} + a_2\frac{d}{dy} + a_3\frac{d}{dz}$

$\therefore (\vec{a}\cdot\nabla)f = \left(a_1\frac{d}{dx} + a_2\frac{d}{dy} + a_3\frac{d}{dz}\right)f$

$= a_1\frac{df}{dx} + a_2\frac{df}{dy} + a_3\frac{df}{dz}$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{j})\left(\frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j} + \frac{df}{dz}\hat{k}\right)$

$= \vec{a}\cdot\nabla f$

(4) $(\vec{a}\cdot\nabla)\vec{r} = \vec{a}$

$\Rightarrow \left(a_1\frac{d}{dx} + a_2\frac{d}{dy} + a_3\frac{d}{dz}\right)(x\hat{i} + y\hat{j} + z\hat{k})$

$\Rightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\Rightarrow \vec{a}$

page-11 (ii) $\text{div}\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$.

$\Rightarrow \text{div}\left(\frac{\vec{r}}{r}\right) = \nabla\cdot\frac{\vec{r}}{r} = \nabla\cdot\frac{\vec{r}}{r}$

$\Rightarrow \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$

$$= \sum \frac{d}{dx}\left(\frac{x}{r}\right) = \sum \left\{1\cdot\frac{1}{r} + x\left(\frac{-1}{r^2}\right)\frac{x}{r}\right\}$$

$\frac{x}{r} = x\cdot\frac{1}{r}$

$$= \sum \frac{1}{r} - \frac{1}{r^3}\sum x^2$$

$$= \frac{3}{r} - \frac{1}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3}{r} - \frac{1}{r^3}(r^2) = \frac{2}{r}$$

अतः $\boxed{\text{div.}(\vec{r}) \text{ या } \nabla\vec{r} = \frac{2}{r}}$

Detail में-

$$\text{div.}(\vec{r}) = \nabla\cdot\vec{r} = \nabla\left(\frac{\vec{r}}{r}\right)$$

$$= \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)\cdot\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= \frac{d}{dx}\left(\frac{x}{r}\right) + \frac{d}{dy}\left(\frac{y}{r}\right) + \frac{d}{dz}\left(\frac{z}{r}\right)$$

$$= \left\{1\cdot\frac{1}{r} + x\left(\frac{-1}{r^2}\right)\frac{x}{r}\right\} + \left\{1\cdot\frac{1}{r} + y\left(\frac{-1}{r^2}\right)\frac{y}{r}\right\} + \left\{1\cdot\frac{1}{r} + z\left(\frac{-1}{r^2}\right)\frac{z}{r}\right\}$$

$$= \frac{3}{r} - \frac{1}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

(iii) $\text{div}(r^n\vec{r}) = (n+3)r^n$

prove- $\text{div}(r^n\vec{r}) = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)\cdot\left\{r^n(x\hat{i}+y\hat{j}+z\hat{k})\right\}$

$$= \sum \frac{d}{dx}(r^n\cdot x)$$

$$= \sum\left\{r^n\cdot 1 + x\cdot n\cdot r^{n-1}\cdot\frac{x}{r}\right\}$$

$$= \sum\left[r^n + n r^{n-2}x^2\right]$$

$$= \sum\left[r^n\right] + n r^{n-2}\sum x^2$$

$$= 3r^n + n r^{n-2}(r^2)$$

$$= 3r^n + n r^n$$

$$= (3+n)r^n$$

(vi) $\nabla\cdot\left(\frac{f(r)\vec{r}}{r}\right) = \frac{1}{r^2}\frac{d}{dr}(r^2 f)$

Proof L.H.S. $= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{\frac{f(r)}{r}(x\hat{i}+y\hat{j}+z\hat{k})\right\}$

$= \sum \frac{\partial}{\partial x}\left\{f(r)\cdot\frac{1}{r}\cdot x\right\}$

$= \sum\left\{\frac{f(r)}{r}\cdot 1 + f(r)x\left(\frac{-1}{r^2}\right)\frac{x}{r} + \frac{f'(r)x}{r}\cdot\frac{x}{r}\right\}$

$= \sum\left[\frac{f(r)}{r} + -\frac{f(r)x^2}{r^3} + \frac{f'(r)x^2}{r^2}\right]$

$= \sum\frac{f(r)}{r} + -\frac{f(r)}{r^3}\sum x^2 + \frac{f'(r)}{r^2}\sum x^2$

$= \sum\frac{f(r)}{r} - \frac{f(r)}{r^3}\cdot r^2 + \frac{f'(r)}{r^2}\cdot r^2$

$= \sum\frac{f(r)}{r} - \frac{f(r)}{r} + f'(r)$

$= 3\frac{f(r)}{r} - \frac{f(r)}{r} + f'(r)$

$= 2\frac{f(r)}{r} + f'(r)$

$\therefore \boxed{\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = f'(r) + 2\frac{f(r)}{r}}$

$= \frac{1}{r^2}\left[r^2 f'(r) + 2rf(r)\right]$

$\boxed{\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = \frac{1}{r^2}\frac{d}{dr}(r^2 f)}$

(2) (iii) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$

Proof L.H.S $= \nabla^2 f(r)$

$= \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) f(r)$

$= \sum\left(\frac{d^2}{dx^2} f(r)\right)$

$$= \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right)$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\cdot \frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left(f'(r)\cdot \frac{1}{r}\cdot x\right)$$

$$= \sum \left[\frac{f'(r)}{r}\cdot 1 + x f'(r)\left(\frac{-1}{r^2}\right)\frac{x}{r} + \frac{x}{r} f''(r)\frac{x}{r}\right]$$

$$= \sum \frac{f'(r)}{r} - \frac{f'(r)}{r^3}\sum x^2 + \frac{f''(r)}{r^2}\sum x^2$$

$$= \frac{3f'(r)}{r} - \frac{f(r)\cdot r^2}{r^3} + \frac{f''(r)}{r^2}\cdot r^3$$

$$\boxed{\nabla^2 f(r) = \frac{2f'(r)}{r} + f''(r)}$$

Q.16] $\because \vec{f}$ एक ~~परिनालिका सदिश~~ होगा यदि $div\vec{f} = 0$

$$\nabla\cdot\vec{f} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[(x+3y)\hat{i} + (x-2z)\hat{j} + (x+az)\hat{k}\right]$$

$$0 = 1 + 0 + a$$

$$\Rightarrow a = -1$$

Q.19] दिक् अवकलज पृष्ठ के लम्बवत् महत्तम होता है।

Q.23] $\vec{f} = x^2z\hat{i} - 2y^2z^2\hat{j} + xy^2z\hat{k}$

$$\nabla\times\vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2y^2z^2 & xy^2z \end{vmatrix}$$

$$= \hat{i}\left[-2y^2(2z) - 2xyz\right] - \hat{j}\left[x^2 + y^2z\right] + \hat{k}\left[0 - 0\right]$$

$$(\nabla\times\vec{f})_{(1,1,-1)} = (-4y^2z + 2xyz)\hat{i} + (x^2 - y^2z)\hat{j}$$

$$= \left[-4(-1) + 2\right]\hat{i} + (1+1)\hat{j}$$

$$= 6\hat{i} + 2\hat{j}$$

Q25) $\nabla \cdot (\nabla \times \vec{f}) \Rightarrow \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x^2y & xz & 2yz \end{vmatrix}$

$= (2z - x)\hat{i} - \hat{j}(0-0) + \hat{k}(z - x^2)$

$\nabla \times \vec{F} = (2z-x)\hat{i} + (z-x^2)\hat{k}$

$\nabla \cdot (\nabla \times \vec{F}) = \left(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{j}\right) \cdot \left[(2z-x)\hat{i} + (z-x^2)\hat{k}\right]$

$= -1 + 1 = 0$

Q27) $\int_0^1 (e^t \hat{i} + e^{-2t}\hat{j} + t\hat{k})\, dt$

$\Rightarrow \int_0^1 \left[e^t - \frac{e^{-2t}}{2} + \frac{t^2}{2}\right]_0^1$

$\Rightarrow \left[e^1 - \frac{e^{-2}}{2} - e^0 + \frac{e^0}{2}\right]$

$\Rightarrow \hat{i}\int_0^1 e^t dt + \hat{j}\int_0^1 e^{-2t} dt + \hat{k}\int_0^1 t\, dt$

$\Rightarrow \hat{i}\left[e^t\right]_0^1 + \hat{j}\left[\frac{e^{-2t}}{-2}\right]_0^1 + \hat{k}\left[\frac{t^2}{2}\right]_0^1$

$\Rightarrow (e^1 - e^0)\hat{i} + \left[\frac{e^{-2}}{-2} + \frac{e^{-0}}{+2}\right]\hat{j} + \left[\frac{1}{2} - 0\right]\hat{k}$

$\Rightarrow (e-1)\hat{i} + \frac{1}{2}(1 - e^{-2})\hat{j} + \frac{1}{2}\hat{k}$

Q30) $\because$ वरण $\vec{a} = \frac{dv}{dt} = t\hat{i} + 2\hat{j} + 3t^2\hat{k}$

Int. w.r.to 't' →

$v = \int (t\hat{i} + 2\hat{j} + 3t^2\hat{k})$

$v = \hat{i}\frac{t^2}{2} + 2t\hat{j} + t^3\hat{k} + c$ — (1)

But $v_{t=0} = 0 \Rightarrow v = 2\hat{i} + \hat{j}$

$2\hat{i} + \hat{j} = c$

Sub. by (1) $\Rightarrow$ $\boxed{v = \left(\frac{t^2}{2} + 2\right)\hat{i} + (2t+1)\hat{j} + t^3\hat{k}}$

Q.31) $\int_0^2 (\vec{r}\cdot\vec{s})\,dt$

$\Rightarrow \int_0^2 [t\hat{i} - t^2\hat{j} + (t-1)\hat{k}]\cdot[2t^2\hat{i} + 6t\hat{k}]\,dt$

$\Rightarrow \int_0^2 (2t^3 + 6t^2 - 6t)\,dt$

$\Rightarrow \left[\frac{2t^4}{4} + \frac{6t^3}{3} - \frac{6t^2}{2}\right]_0^2$

$\Rightarrow \left[\frac{1}{2}(16) + 2(8) - 3(4)\right]$

$\Rightarrow [8 + 16 - 12] = 12$

Q.32) $\int_0^2 (\vec{r}\times\vec{s})\,dt$

$$\vec{r}\times\vec{s} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -t^2 & (t-1) \\ 2t^2 & 0 & 6t \end{vmatrix}$$

$= (-6t^3 - 0)\hat{i} - \hat{j}(6t^2 - 2t^3 + 2t^2) + \hat{k}(2t^4)$

$\int \vec{r}\times\vec{s} = \int_0^2 [(-6t^3)\hat{i} - (8t^2 - 2t^3)\hat{j} + (2t^4)\hat{k}]\,dt$

$= \hat{i}\left[-\frac{6t^4}{4}\right]_0^2 - \left(\frac{8t^3}{3} - \frac{2t^4}{4}\right)_0^2\hat{j} + \left(\frac{2t^5}{5}\right)_0^2\hat{k}$

$= -\frac{3}{2}(16)\hat{i} - \left[\frac{8}{3}(8) - \frac{1}{2}(16)\right]\hat{j} + \frac{2}{5}(32)\hat{k}$

$= -24\hat{i} - \left[\frac{64}{3} - 8\right]\hat{j} + \frac{64}{5}\hat{k}$

$= -24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}$

Q.33) $\vec{F} = (x^2 - y^2)\hat{i} + xy\hat{j}$ (0,0) to (2,8)

& curve $y = x^3$ — (1)

$x = 0$ to $x = 2$

$y = 0$ to $y = 8$

$\Rightarrow \int_C F\cdot dr = \int_C [(x^2 - y^2)\hat{i} + xy\hat{j}]\cdot[dx\hat{i} + dy\hat{j} + dz\hat{k}]$

$= \int_C (x^2 - y^2)\,dx + \int_C xy\,dy$

Now put the Value from (1) ⇒

$$\Rightarrow \int_0^2 (x^2 - x^6)\,dx + \int_0^8 y^{1/3} y\,dy$$

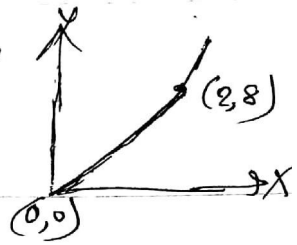
$$\Rightarrow \left[\frac{x^3}{3} - \frac{x^7}{7}\right]_0^2 + \left[\frac{3}{7} y^{7/3}\right]_0^8$$

$$\Rightarrow \left[\frac{8}{3} - \frac{8\times8\times2}{7}\right] + \frac{3}{7}(8)^{7/3}$$

$$\Rightarrow \left(\frac{8}{3} - \frac{128}{7}\right) + \frac{3}{7}\left[(2)^3\right]^{7/3}$$

$$\Rightarrow \left(\frac{56-384}{21}\right) + \frac{3}{7}(2^7)$$

$$\Rightarrow -\frac{328}{21} + \frac{128\times3}{7} = \frac{-328+1152}{21} = \frac{824}{21}$$



Q.51) $\int_C \vec{F}\cdot d\vec{r}$ given $y = x^3$ — (1)

$x = 0$ to $x = 2$

$y = 0$ to $y = 8$

$$\Rightarrow \int_C \left[(x^2 - y^2)\hat{i} + xy\hat{j}\right]\left[dx\hat{i} + dy\hat{j}\right]$$

$$\Rightarrow \int_C \left[(x^2 - y^2)\,dx + xy\,dy\right]$$

put Value from (1) →

$$\Rightarrow (y^{1/3} - y^2)\,dx$$

$$\Rightarrow \int_0^2 (x^2 - x^6)\,dx + \int_0^8 y^{1/3} y\,dy$$

$$\Rightarrow \left(\frac{x^3}{3} - \frac{x^7}{7}\right)_0^2 + \frac{3}{7} y^{7/3} + c$$

$$\Rightarrow \left(\frac{8}{3} - \frac{8\times8\times2}{7}\right) + \frac{3}{7}(2^3)^{7/3}$$

$$\Rightarrow \left(\frac{8}{3} - \frac{128}{7}\right) + \frac{3}{7}(8\times8\times2) = \frac{824}{21}$$

Q.43] $\int_C \vec{F}\cdot d\vec{r}$ given $y = x^2 - 4$

$x = 2$ to $x = 4$

$y = 0$ to $y = 12$

$$\Rightarrow \int xy\,dx + \int_C (x^2 + y^2)\,dy$$

$$\Rightarrow \int_2^4 x(x^2 - 4)\,dx + \int_0^{12} (y + 4 + y^2)\,dy$$