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# RRB - JE

# MECHANICAL

## Railway Recruitment Board

Volume - 9

## Heat and Mass Transfer

SIERRA INNOVATIONS

# CONTENTS

# HEAT CONDUCTION

## THEORY

General relation for Fourier Law of heat conduction,

$$\dot{Q}_n = -KA\frac{\partial T}{\partial n}$$

Where n = normal of isothermal surface at point P.

In rectangular co-ordinates, the heat conduction vector can be expressed in terms of its components as

$$\vec{\dot{Q}}_n = \vec{\dot{Q}}_x\hat{i} + \dot{Q}_y\hat{j} + \dot{Q}_z\hat{k}$$

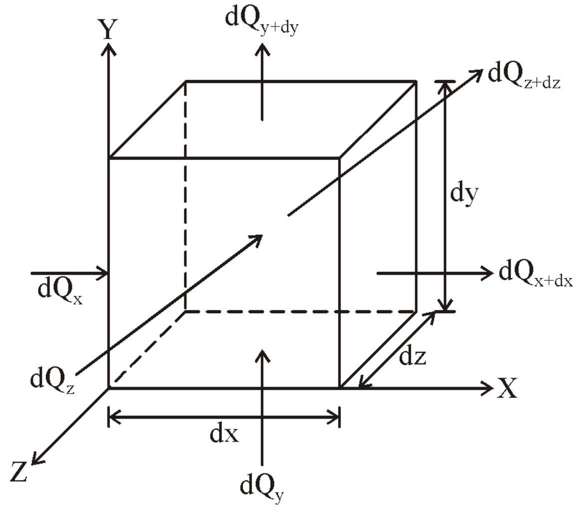
Where, $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors and $\dot{Q}_x, \dot{Q}_y$ and $\dot{Q}_z$ are the magnitude of heat transfer rates in x, y and z-directions which can be determined by Fourier Law as

$$\dot{Q}_x = -KA_x\frac{\partial T}{\partial x}, \quad \dot{Q}_y = -KA_y\frac{\partial T}{\partial y}, \quad \dot{Q}_z = -KA_z\frac{\partial T}{\partial z}$$

### 2.1 General Heat Conduction Equation

Let us consider an infinitesimal volume element of side dx, dy and dz.

Now consideration here will include the non steady condition of temperature variation with time t.




According to Fourier heat conduction law, the heat flowing into the left most face of the element in the X-direction.

$$dQ_x = -K.dy.dz.\frac{\partial T}{\partial x}$$

From Taylor's series

$$dQ_{x+dx} = dQ_x + \frac{\partial}{\partial x}(dQ_x).dx$$

The net heat flow by conduction in X-direction.

$$dQ_x - dQ_{x+dx} = -\frac{\partial}{\partial x}(dQ_x).dx$$

$$= -\frac{\partial}{\partial x}\left(-K.dy.dz.\frac{\partial T}{\partial x}\right).dx$$

$$= K.dx.dy.dz.\frac{\partial^2 T}{\partial x^2} \qquad ...(1)$$

**Similarly in Y-direction and z-direction**

$$dQ_y - dQ_{y+dy} = K.dx.dy.dz.\frac{\partial^2 T}{\partial y^2} \qquad ...(2)$$

$$dQ_z - dQ_{z+dz} = K.dx.dy.dz.\frac{\partial^2 T}{\partial z^2} \qquad ...(3)$$

Let $q_g$ is the rate at which heat is generated initially per unit volume.
Then the total rate of heat generation in elemental volume is = $q_g$.dxdy.dz ...(4)

The rate of accumulation of internal energy within the control volume = $mC\frac{\partial T}{\partial t}$

$$= \rho.dxdy.dz\frac{\partial T}{\partial t} \qquad ...(5)$$

**From energy balance equation**

Rate of energy storage within the solid = Rate of heat influx – Rate of heat outflux + Rate of heat generation

$$\Rightarrow \quad \rho c.dxdy.dz.\frac{\partial T}{\partial t} = (dQ_x + dQ_y + dQ_z) - (dQ_{x+dx} + dQ_{y+dy} + dQ_{z+dz}) + q_G.dxdy.dz$$

$$\Rightarrow \quad \rho.C.\frac{\partial T}{\partial t} = K\left\{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right\} + q_G$$

$$\Rightarrow \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_G}{K} = \frac{\rho C}{K}.\frac{\partial T}{\partial t}$$

$$\Rightarrow \quad \nabla^2 T + \frac{q_G}{K} = \frac{1}{\alpha}.\frac{\partial T}{\partial t}$$

Where $\alpha \rightarrow$ thermal diffusivity

**In cylindrical co-ordinate**

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r.\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}.\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_G}{K} = \frac{1}{\alpha}.\frac{\partial T}{\partial t}$$

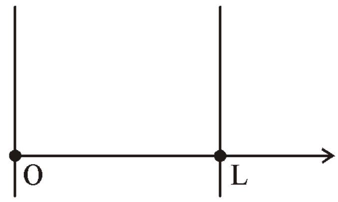
**In spherical co-ordinate**

$$\frac{1}{r^2}.\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2 \sin\theta}.\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}+\frac{q_G}{K} = \frac{1}{\alpha}.\frac{\partial T}{\partial t}$$

### *2.1.1 Boundary and Initial Condition*

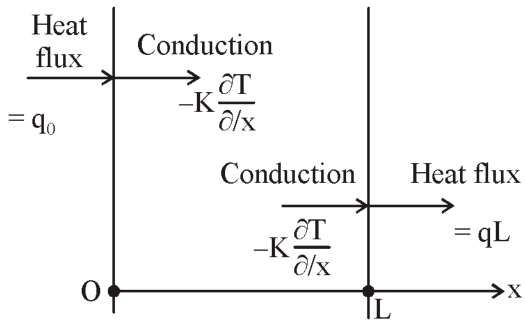
To describe a heat transfer problem completely, two boundary condition must be given for each direction of coordinate system along which heat transfer is significant.

**(i) Specified temperature boundary condition :** (Drichilet Boundary Condition)




$$T(0,\ t) = T_1 \text{ and } T(L,\ t) = T_2$$

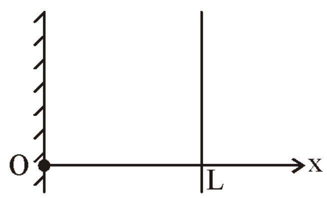
**(ii) Specified heat flux boundary condition :** For a plate of thickness L subjected to heat flux of 50 $W/m^2$ into medium from both side, the specified flux boundary conditions are




$$-K\frac{\partial T}{\partial x}(L,t) = -\ 50$$

Since flux at surface x = L is in negative x-direction, thus it is – 50 $W/m^2$.

**Special case (1) :** Insulated boundary

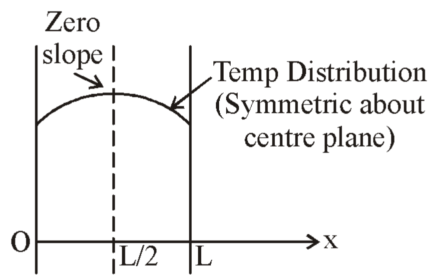



$$\dot{q} = 0 \Rightarrow -K\frac{\partial T}{\partial x}(0,t) = 0$$

$$\Rightarrow \quad \frac{\partial T}{\partial x}(0,t) = 0$$

That is, on an insulated surface, the first derivative of temperature w.r.t. space variable (the temperature gradient) in direction normal to insulated surface is zero. This means temperature function must be perpendicular to insulated surface since slope of temperature at surface must be zero.

**Special case (2) :** Thermal Symmetry



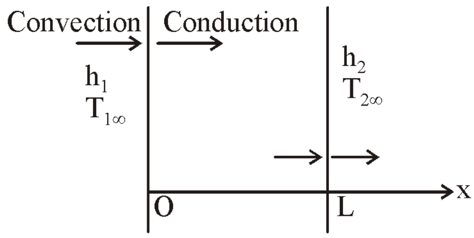

It possesses thermal symmetry about centre plane at

$$x = \frac{L}{2}$$

$$\frac{\partial T}{\partial x}\left(\frac{L}{2}, t\right) = 0$$

**(iii) Convectional boundary condition :** The convection boundary condition is based on surface energy balance

Heat conduction at surface in a selected direction = Heat convection at surface in the same direction

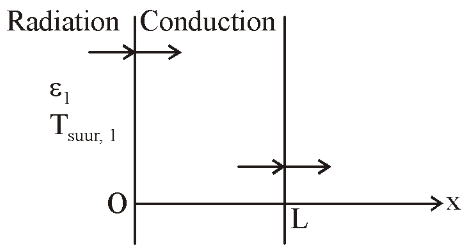



$$-K\frac{\partial T(0,t)}{\partial t} = -\ h_1\left[T_{\infty 1} - T_{(0,t)}\right]$$

and $$-K\frac{\partial T(L,t)}{\partial t} = h_1\left[T(L,t) - T_{\infty 2}\right]$$

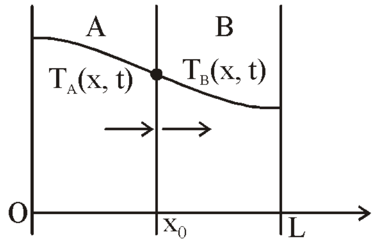
**(iv) Radiation boundary condition :**

Heat conduction at surface in a selected direction = Radiation exchange at surface in the same direction




$$-K\frac{\partial T(0,t)}{\partial x} = \varepsilon_1\sigma\left[T^4_{surr,1} - T^4_{(0,t)}\right]$$

**(v) Interface boundary condition :** The boundary condition at interface are based on requirement




(a) Two bodies in contact must have same temperature at area of contact.

(b) An interface cannot store energy, thus heat flux on both side of interface must be same.

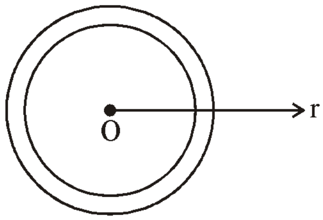
$$T_A\,(x_A,\,t) = T_B\,(x_0,\,t)$$

$$-K_A \frac{\partial T_A(x_0,t)}{\partial x} = -K_B \frac{\partial T_B(x_0,t)}{\partial n}$$

***Example 1 :*** Consider a spherical container of inner radius $r_1$ = 8 cm and $r_2$ = 10 cm, K = 45 W/m°C. The inner and outer surface of container are maintained at temperature of $T_1$ = 200°C and $T_2$ = 80°C. Determine general relation for temperature distribution inside the shell under steady condition and determine the rate of heat loss from the container.

***Solution :***

Heat transfer is steady the heat transfer is one-dimensional since there is thermal symmetry about mid point.




$$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

So isotherm are concentric sphere. So T = T(r)

Heat conduction equation,

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0$$

$$\frac{dT}{dr} = \frac{c_1}{r^2}$$

$$T(r) = \frac{c_1}{r} + c_2$$

Boundary condition are,

$$T(r) = T_1 = 200°C$$

$$T(r_2) = T_2 = 80°C$$

$$T_1 = \frac{-c_1}{r_1} + c_2 \text{ and } T_2 = \frac{-c_1}{r_2} + c_2$$

Solving for $c_1$ and $c_2$ and substituting for T,

$$T(r) = \frac{r_1 r_2}{r(r_2 - r_1)}(T_1 - T_2) + \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1}$$

The rate of heat loss from container is rate of heat conduction through container wall,

$$\dot{Q}_{shpere} = -kA\frac{dT}{dr} = +k(4\pi r^2).\frac{r_1 r_2 (T_1 - T_2)}{r^2(r_2 - r_1)} = \frac{4\pi k r_1 r_2 (T_1 - T_2)}{r_2 - r_1}$$

## 2.2 Heat Generation in a Solid

Heat generation is expressed per unit volume of medium

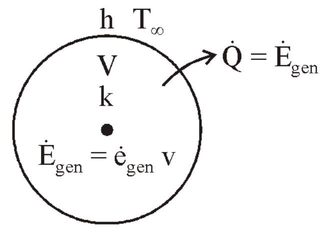
$$\dot{e}_{gen} = \frac{\dot{E}_{gen,electric}}{V_{wire}} = \frac{I^2 Re}{\pi r_0^2 L}$$

The temperature of medium rises during heat generation at start up condition. At steady state, the rate of heat generation equals the rate of heat transfer to surroundings.

The maximum temperature $T_{max}$ in a solid that involves uniform heat generation occur at a location farthest away from outer surface when outer surface of solid is maintained at constant temperature $T_s$. For example, maximum temperature occur at mid plane in sphere. The temperature distribution within solid in these cases is symmetrical about centre of symmetry.

Consider a solid medium of surface area $A_s$ and volume V and volume V and constant thermal conductivity, where heat is generated at $\dot{e}_{gen}$ per unit volume under steady condition.

Rate of heat transfer from the solid= Rate of heat generation within the solid




$$\dot{Q} = \dot{e}_{gen} v$$

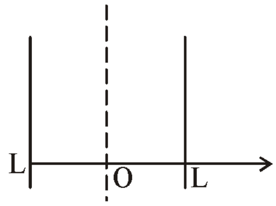
$$\Rightarrow \quad hA_s (T_s - T_\infty) = \dot{e}_{gen} v$$

$$\Rightarrow \quad T_s = T_\infty + \frac{\dot{e}_{gen} v}{hA_s}$$

For large plane wall, of thickness 2L

$$A_s = 2 A_{wall}$$

and $$v = 2 A_{wall} L$$

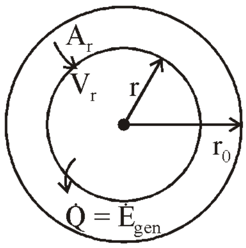



$$T_{s,\text{ plane wall}} = T_\infty + \frac{\dot{e}_{gen}L}{h}$$

For cylinder, $A_s = 2\pi r_0 L,\ v = \pi R_0^2 L$

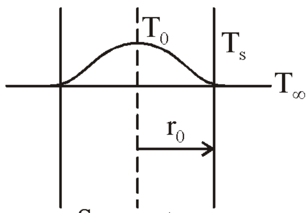
$$T_{s,\text{ cyl}} = T_\infty + \frac{\dot{e}_{gen} r_0}{3h}$$

Consider heat transfer from long solid cylinder. The heat generated within this inner cylinder must be equal to heat conducted through outer surface.




$$-kA_r \frac{dT}{dr} = \dot{e}_{gen} V_r$$

Where $dT = -\dfrac{\dot{e}_{gen}}{2k} r\, dr$




Integrating At $r = 0,\ T = T_0$

$r = r_0,\ T = T_s$

$$T_s - T_0 = -\frac{\dot{e}_{gen}}{2k}\frac{r_0^2}{2} = -\frac{\dot{e}_{gen} r_0^2}{4k}$$

$\Rightarrow$ $$T_s - T_0 = \frac{\dot{e}_{gen} r_0^2}{4k}$$

$\Rightarrow$ $$\Delta T_{max} = T_0 - T_s = \frac{\dot{e}_{gen} r_0^2}{4k}$$

For plane wall, $$\Delta T_{max} = \frac{\dot{e}_{gen} L^2}{2k}$$

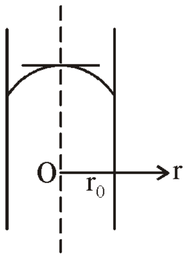
For sphere, $$\Delta T_{max} = \frac{\dot{e}_{gen} r_0^2}{6k}$$

***Example 2 :*** A long homogenous resistance wire of radius $r_0 = 0.5$ cm and thermal conductivity k = 13.5 W/m°C is being used to boil water at atmosphere pressure by passage of electric current. Heat is generated in wire uniform *l*y as result of resistance heating at the rate of $\dot{e}_{gen} = 4.3 \times 10^7$ W/m$^3$. If the outer surface temperature of wire is measured to be $T_s = 180°C$, obtain relation for temperature distribution, and determine temperature at centre line of wire when steady operating condition are reached.

***Solution :***

Heat conduction equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0$$




Two boundary condition are

$$\frac{dT(0)}{dr} = 0,\ T(r_0) = T_s = 108°C$$

Integrating above equation

$$r\frac{dT}{dr} = -\frac{\dot{e}_{gen} r^2}{2k} + c_1$$

$$T = -\frac{\dot{e}_{gen} r^2}{4k} + c_1 \text{In } r + c_2$$

Applying first condition $$c_1 = 0$$

$$T = T_s + \frac{\dot{e}_{gen}}{4k}\left(r_0^2 - r^2\right)$$

Maximum temperature occur at centre line r = 0

$$T = T_s + \frac{\dot{e}_{gen}}{4k} r_0^2$$

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