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RRB - JE

ELECTRONICS

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Volume - 6

Communication

SIERRA INNOVATIONS

CONTENTS

FOURIER SERIES AND FOURIER TRANSFORM

THEORY

1. FOURIER SERIES :

A fourier series is a representation of a function f(t) by the linear combination of elements of a complete set of infinite mutually orthogonal functions.

These elements must be mutuallly orhtogonal.

Note: **(i) Mutually orthogonal functions:**

Two functions are said to be mutually orthogonal over an interval between t_1 and t_2, if the integral of their product over this interval is zero.

i.e. $$\int_{t_1}^{t_2} f(t)h(t) = 0$$

In general $$\int_{t_1}^{t_2} f_i(t)f_k(t)dt = 0 \qquad (i \neq k)$$

(ii) It assure that one function f(t) does not have any component of other function g(t).

(iii) The example of orthogonal functions are: Lengendre polynomial, Jacobi polynomials, Trigonometric and exponential function.

(iv) Orthogonalities in complex functions

$$\int_{t_1}^{t_2} f_1(t)f_2^*(t)dt = \int_{t_1}^{t_2} f_1^*(t)f_2(t)dt = 0$$

where, f_1^* and f_2^* are complex conjugate of $f_1(t)$ and $f_2(t)$ respectively.

2. DIRICHLET'S CONDITIONS

There are sufficient conditions that needs to be satisfied by a function f(t) for its fourier series representation within the interval (t_1, t_2). These are follows:

(i) x(t) is absolutely integrable, i.e. $\int_{-\infty}^{+\infty} |x(t)|dt < \infty$

(ii) x(t) is single valued and has only finite number of maxima and minima within any finite interval.

(iii) x(t) has a finite number of finite discontinuities within any finite interval.

Note: *Fourier series is valid for periodic signals only.*

3. TRIGONOMETRIC FOURIER SERIES :

A function f(t) can be represented by a fourier series comprising the following sine and cosine functions:

$$f(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t)$$

$$+ \ldots\ldots + a_n\cos(n\omega_0 t) + \ldots\ldots + b_1\sin(\omega_0 t)$$

$$+ b_2\sin(2\omega_0 t) + \ldots\ldots + b_n\sin(n\omega_0 t) + \ldots\ldots$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right],\ t_0 \le t \le t_0 + T$$

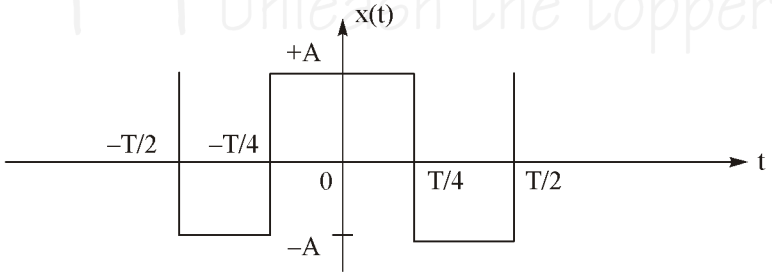
Where $T = \dfrac{2\pi}{\omega_0}$

$$a_0 = \frac{1}{T}\int_{\text{Over a one time period}} f(t)\,dt = \text{D.C. or average value}$$

$$a_n = \frac{2}{T}\int_{\text{over an one time period}} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{\text{over an one time period}} f(t)\sin(n\omega_0 t)\,dt$$

Example: Find the trigonometric fourier series representation of following figure



Solution:

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right] \quad \ldots(i)$$

Signal x(t) is given by

$$x(t) = \begin{cases} -A & ; -\dfrac{T}{2} < t < -\dfrac{T}{4} \\ +A & ; -\dfrac{T}{4} < t < +\dfrac{T}{4} \\ -A & ; \dfrac{T}{4} < t < \dfrac{T}{2} \end{cases}$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt$$

$$a_n = \frac{2}{T}\left[\int_{-\frac{T}{2}}^{-\frac{T}{4}} -A\cos(n\omega_0 t)\,dt + \int_{-\frac{T}{4}}^{\frac{T}{4}} A\cos(n\omega_0 t)\,dt + \int_{\frac{T}{4}}^{\frac{T}{2}} -A\cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2A}{T}\left[-\int_{-\frac{T}{2}}^{-\frac{T}{4}} \cos(n\omega_0 t)\,dt + \int_{-\frac{T}{4}}^{+\frac{T}{4}} \cos(n\omega_0 t)\,dt - \int_{\frac{T}{4}}^{\frac{T}{2}} \cos(n\omega_0 t)\,dt\right]$$

$$a_n = \frac{2A}{T}\left[\left[-\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_{-\frac{T}{2}}^{-\frac{T}{4}} + \left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_{-\frac{T}{4}}^{+\frac{T}{4}} + \left[-\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_{\frac{T}{4}}^{\frac{T}{2}}\right]$$

By solving this, we get

$$a_n = \frac{8A}{n\omega_0 T}\sin\left(\frac{n\omega_0 T}{4}\right) - \frac{4A}{n\omega_0 T}\sin\left(\frac{n\omega_0 T}{2}\right)$$

$$a_n = \frac{8A}{2n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4A}{2n\pi}\sin(n\pi) \quad ; \quad \omega_0 T = 2\pi$$

$$a_n = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right) - 0$$

The second term in above expression is zero for all integer values of n.

$$a_n = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

and $b_n = 0$ because given waveform is even function. and $a_0 = 0$ because given wave form is symmetrical about horizontal axis.

Putting the values of a_n in equation (i)

$$x(t) = \frac{4A}{\pi}\left[\cos(\omega_0 t) - \frac{1}{3}\cos(3\omega_0 t) + \frac{1}{5}\cos(5\omega_0 t).....\right]$$ ***Ans.***

Note: 1. Symmetry conditions

(i) If x(t) = even function ; then, $b_n = 0$.

(ii) If x(t) = odd function ; then $a_0 = 0$ and $a_n = 0$.

(iii) If x(t) is symmetrical about horizontal axis, then $a_0 = 0$.

2. Some trigonometric identity

(i) $\sin\left(\frac{n\pi}{2}\right) = \begin{matrix} -1; & n = 3,7,11 \\ +1; & n = 1,5,9 \\ 0; & n = \text{even} \end{matrix}\Bigg] n = 0,1,2,3,.....$

(ii) $\cos\left(\frac{n\pi}{2}\right) = \begin{matrix} 0; & n = \text{odd} \\ (-1)^{n/2}; & n = \text{even} \end{matrix}\Bigg] n = 0,1,2,3,4,.....$

(iii) $\tan\left(\frac{n\pi}{2}\right) = \infty$; n = 1,2,3,.....

(iv) $\sin(n\pi) = 0$; n = 0,1,.....

(v) $\cos(n\pi) = (-1)^n$; n = 0,1,.....

(vi) $\tan(n\pi) = 0$; n = 0,1,.....

4. POLAR FOURIER SERIES REPRESENTATION :

A function x(t) can be represented by a polar fourier series

$$x(t) = D_0 + \sum_{n=1}^{\infty} D_n\left(n\omega_0 t - \phi_n\right)$$

where, $D_0 = a_0 = \frac{1}{T}\int_{\text{over a time period}} x(t)\,dt$ = D.C. value or average value.

$$D_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

5. COMPLEX / EXPONENTIAL FOURIER SERIES

A function x(t) with period T can be represented by complex / exponential fourier series.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \exp\left(jn\omega_0 t\right) \qquad ...(ii)$$

where, $\omega_0 = \frac{2\pi}{T}$

$$C_n = \frac{1}{T}\int_{\text{over a time period}} x(t)\exp\left(-jn\omega_0 t\right)dt$$

Coefficient C_n are in general complex form, hence

$$C_n = |C_n| \exp(j\phi_n) \quad \text{...(iii)}$$

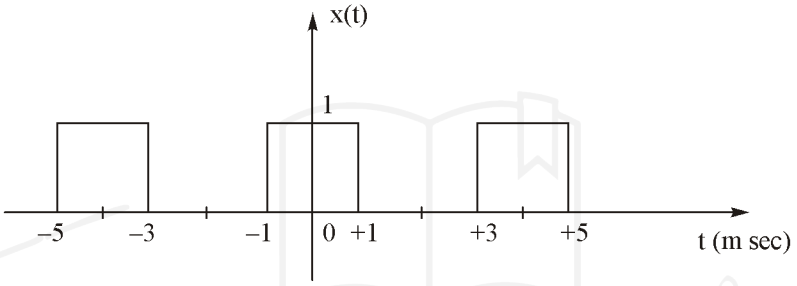
So, using equation (iii) in equation (ii)

$$x(t) = \sum_{n=-\infty}^{\infty} |C_n| \exp\left(j\left(n\omega_0 t + \phi_n\right)\right)$$

The plot between $|C_n|$ versus n (or $n\omega_0$) is called magnitude spectrum and ϕ_n versus n (or $n\omega_0$) is called phase spectrum.

It is important to note that the spectrum of a periodic signal exists only at discrete frequency.

Example: For the unit amplitude reactangular pulse train shown in figure below, compute the fourier series coefficient.



Solution: Signal x(t) has a period T = 4 millisecond and it is ON for half the period and OFF during the remaining half.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_0 t)dt$$

$$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$$

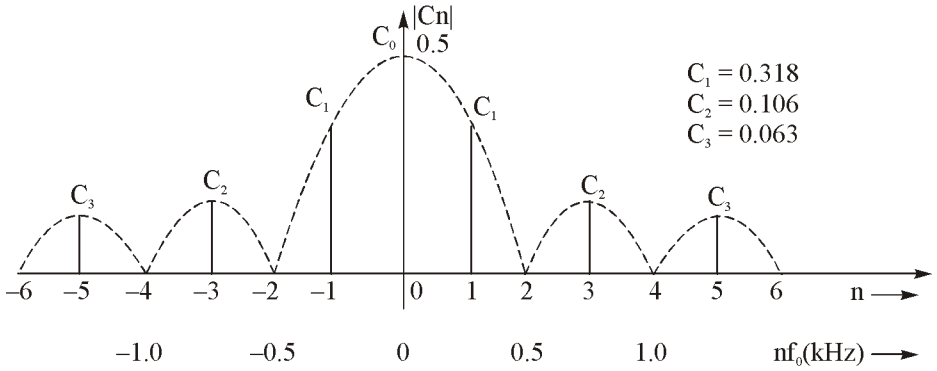
$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)dt$$

$$= \frac{1}{T}\int_{-T/4}^{T/4} \exp(-jn\omega_0 t)dt$$

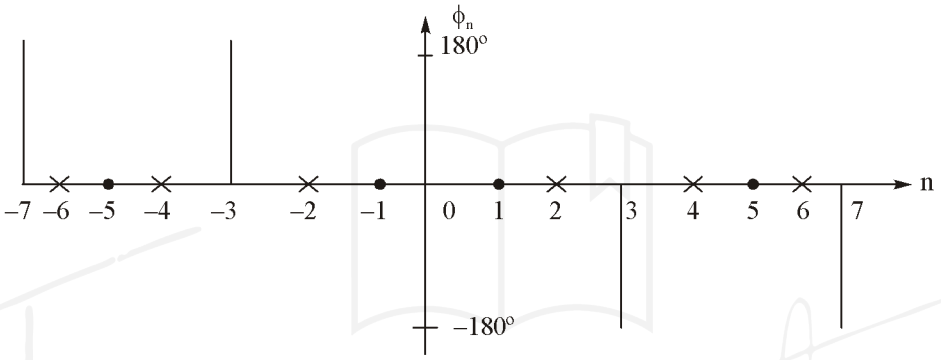
$$= \frac{1}{T}\frac{2\sin\left(\frac{n\pi}{2}\right)}{n\omega_0}$$

$$= \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

All the fourier coefficients are real but could be positive and negative. Hence ϕ_n is either zero or $\pm\pi$ for all n.



(a) Megnitude spectrum.



(b) Phase spectrum.

- C_0, the vaerage or the DC value of pulse train is $\frac{1}{2}$.
- Spectrum exists only at discrete frequencies, $f = nf_0$ with f_0 = 250 Hz. Such a spectrum is called the discrete spectrum
- The envelope consists of several lobes and the magnitude of each lobe keeps decreasing with increase in frequency.
- The magnitude spectrum is symmetric and phase spectrum is antisymmetric. This is because x(t) is real.
- ϕ_n at n = ±2, ±4 etc. is undefined at $|C_n| = 0$ for these n. This is indicated with a cross on the phase spectrum plot.

6. SINC FUNCTION

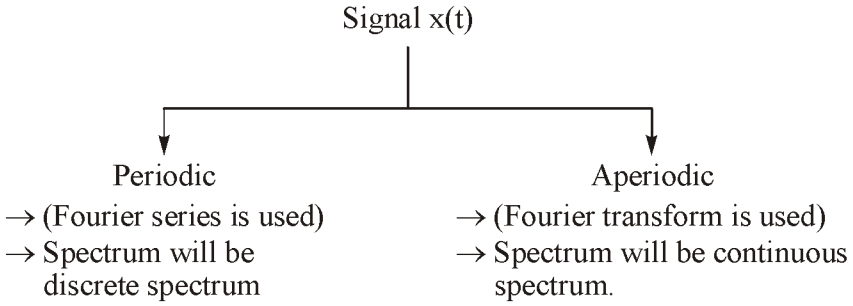
$$\text{sin c } \lambda = \text{sin c } (\lambda) = \frac{\sin(\pi\lambda)}{(\pi\lambda)}$$

7. SAMPLING FUNCTION

$$S_a(\lambda) = \frac{\sin\lambda}{\lambda}$$

8. FOURIER TRANSFORM

Fourier transform is used to find the frequency component in time domain signal.



Fourier transform of x(t) is X(f) . X(f) is defined as

$$x(t) \rightleftharpoons X(f)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \qquad \text{(Analysis equation)}$$

$$X(f) = F\left[x(t)\right]$$

X(f) is, in general, a complex quantitiy.

$$X(f) = X_R(f) + jX_I(f)$$

$$= |X(f)| e^{j\phi(f)}$$

where, $X_R(f)$ = Real part of X(f)

$X_I(f)$ = Imaginary part of X(f)

$|X(f)|$ = Magnitude of X(f)

$$= \sqrt{(X_R(f))^2 + (X_I(f))^2}$$

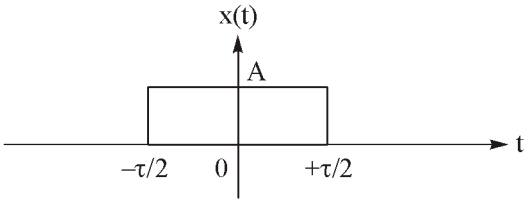
$$\theta(f) = \arg\left[X(f)\right] = \tan^{-1}\left[\frac{X_I(f)}{X_R(f)}\right]$$

The plot between |X(f)| versus f, is known as magnitude spectrum and θ(f) versus f, is known as the phase spectrum.

Inverse fourier transform (IFT) is defined as

$$x(t) = F^{-1}\left[X(f)\right] = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} dt \qquad \text{(Synthesis equation)}$$

Example: Find the fourier transform of figure.



Solution: $x(t) \rightleftharpoons X(f)$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$= \int_{-\tau/2}^{+\tau/2} A \cdot e^{-j2\pi ft}dt$$

$$= A\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{-\tau/2}^{\tau/2}$$

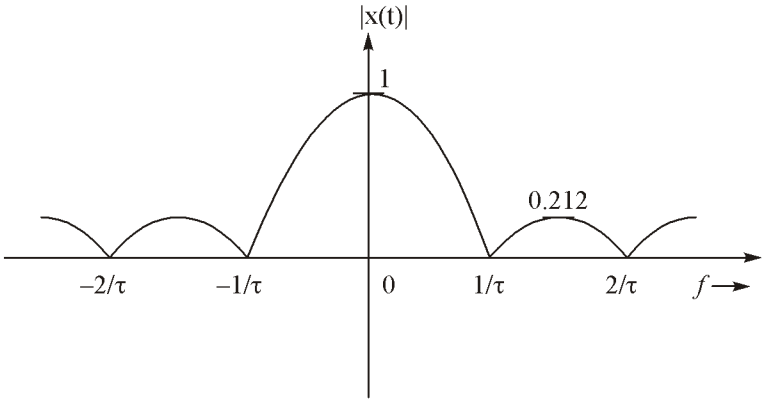
$$= A\tau \ \text{sinc}(\tau f)$$

Note:

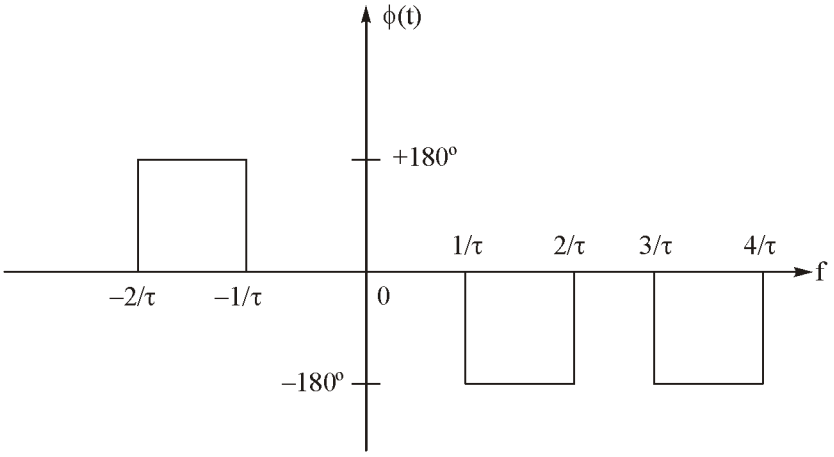
$$\text{sinc}[x] = \frac{\sin(\pi x)}{\pi x} = \begin{cases} 1 ; & x = 0 \\ 0 ; & x = \pm 1, \pm 2, ... \end{cases}$$

$$\text{sinc}(f\tau) = \begin{cases} 1 ; & f\tau = 0 \Rightarrow f = 0 \\ 0 ; & f\tau = \pm 1, \pm 2, \pm 3, ... \\ & f = \pm\frac{1}{\tau}, \pm\frac{2}{\tau}, \pm\frac{3}{\tau}, ... \end{cases}$$

$$X(f) = |X(f)|e^{j\theta(f)}$$



(a) Magnitude spectrum

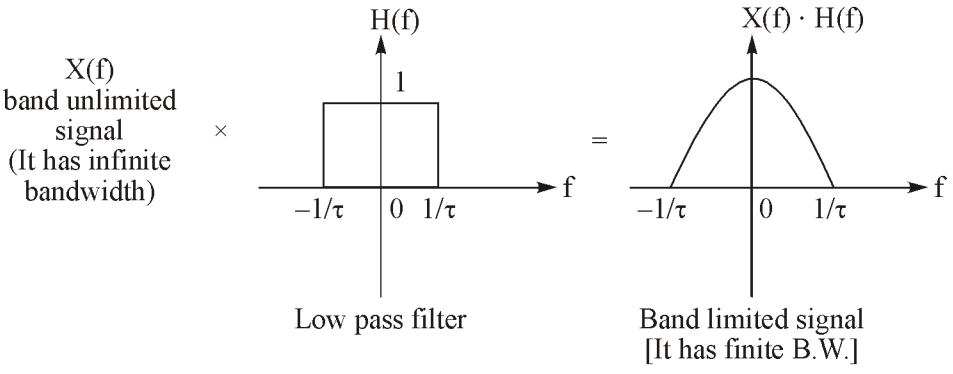


(b) Phase spectrum

- The fourier transform of x(t), X(f) contains all the possible frequencies.
- During the interval, $\frac{m}{\tau} < |f| < \frac{m+1}{\tau}$, with m odd, $\sin c(f\tau)$ is negative. As the magnitude spectrum is always positive, negative value of $\sin c(f\tau)$ are taken care of by making $\theta(f) = \pm 180^{o}$.
- Signal bandwidth (B.W) = Highest positive frequency – Lowest positive frequency

  $= \infty - 0 = \infty$

  for proper transmission of a signal, channel B.W. > signal B.W.
- Signal B.W. is ∞. So, it is not possible to transmit the signal. Signal B.W. should be finite.
- Before transmission, the above signal should be band limited by band limiting process.
- To band limit a signal, all the its significant frequency components has to be retained and insignificant frequency component has to be eliminated.
- Significant frequency contains almost of 95% to 99% of total strength of given signal.



Low pass filter

Band limited signal
[It has finite B.W.]

$$\text{B.W.} = \frac{1}{\tau}$$

$$\tau \neq 0$$

For transmission, significant frequency are given high importance for effective utilization of available channel bandwidth.

9. PROPERTIES OF FOURIER TRANSFORM

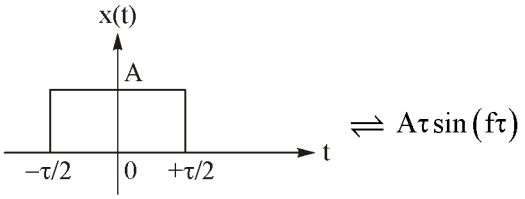
(i) Duality property :

If $x(t) \rightleftharpoons X(f)$

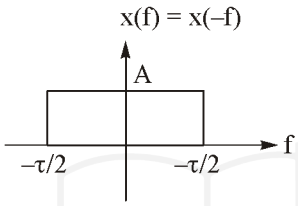
then, $X(t) \rightleftharpoons x(-f)$

Example: Find out the fourier transform of sin c (100t).

Solution :



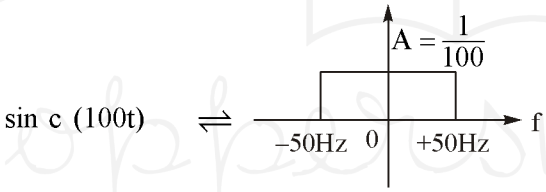
$\rightleftharpoons A\tau \sin(f\tau)$



$A\tau \sin(t\tau) \rightleftharpoons$ (By using duality)

Put $\tau = 100$

and $A = 1$



Example: Find out the fourier transform of $\delta(t)$.

Solution:

$$F[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t) e^{-j2\pi ft} dt$$

$$F[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t) dt \qquad [x(t)\delta(t) = x(0)\delta(t)]$$

$$F[\delta(t)] = 1 \qquad \left[\int_{-\infty}^{\infty} \delta(t)\, dt = \text{Area under curve} = 1\right]$$

$$\delta(t) \rightleftharpoons 1$$

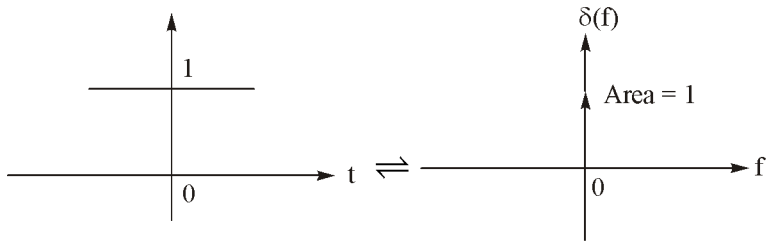
Example: Find the fourier transform of 1.

Solution:

$\delta(t) \rightleftharpoons 1$

$1 \rightleftharpoons \delta(-f)$ [by using duality property]

$1 \rightleftharpoons \delta(f)$ [$\delta(f)$ is an even function]



Containing single frequency component

(ii) Frequency Shifting Property :

If $x(t) \rightleftharpoons X(f)$

then, $x(t)e^{j2\pi f_0 t} \rightleftharpoons X(f - f_0)$

and $x(t)e^{-j2\pi f_0 t} \rightleftharpoons X(f + f_0)$

Example: Find out the fourier transform of $e^{j2\pi f_0 t}$ and $e^{-j2\pi f_0 t}$.

Solution:

$$1 \rightleftharpoons \delta(f)$$

$$1 \cdot e^{j2\pi f_0 t} \rightleftharpoons \delta(f - f_0)$$

$$1 \cdot e^{-j2\pi f_0 t} \rightleftharpoons \delta(f + f_0)$$ [by using frequency shifting property]

$$F\left[e^{j2\pi f_0 t}\right] = \delta(f - f_0)$$

$$F\left[e^{-j2\pi f_0 t}\right] = \delta(f + f_0)$$

Example: Find the fourier transform of $A\cos(2\pi f_0 t)$.

Solution.

$$e^{j2\pi f_0 t} \rightleftharpoons \delta(f - f_0)$$

$$e^{-j2\pi f_0 t} \rightleftharpoons \delta(f + f_0)$$

By adding both equation

$$e^{j2\pi f_0 t} + e^{-j2\pi f_0 t} \rightleftharpoons \delta(f - f_0) + \delta(f + f_0)$$

$$2\cos(2\pi f_0 t) \rightleftharpoons \delta(f - f_0) + \delta(f + f_0)$$

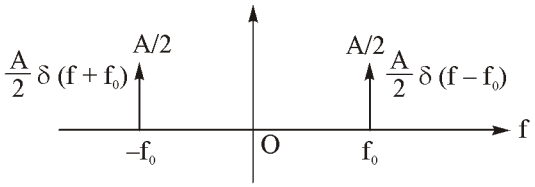
$$\cos(2\pi f_0 t) \rightleftharpoons \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

By multiplying A on both side,

$$A\cos(2\pi f_0 t) \rightleftharpoons \frac{A}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

$$A\cos(2\pi f_0 t) \rightleftharpoons \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0)$$

$A\cos(2\pi f_0 t) \rightleftharpoons$



Fourier transform of $A\cos(2\pi f_0 t)$ has two frequency component at $+f_0$ and $-f_0$.

Example: Find the fourier transform of $x(t)\cos(2\pi f_0 t)$ when fourier transform of $x(t)$ is X(f).

Solution: Given that $x(t) \rightleftharpoons X(f)$

$$x(t)e^{j2\pi f_0 t} \rightleftharpoons X(f-f_0) \qquad \text{[by using shifting property]}$$

$$x(t)e^{-j2\pi f_0 t} \rightleftharpoons X(f+f_0)$$

By adding both,

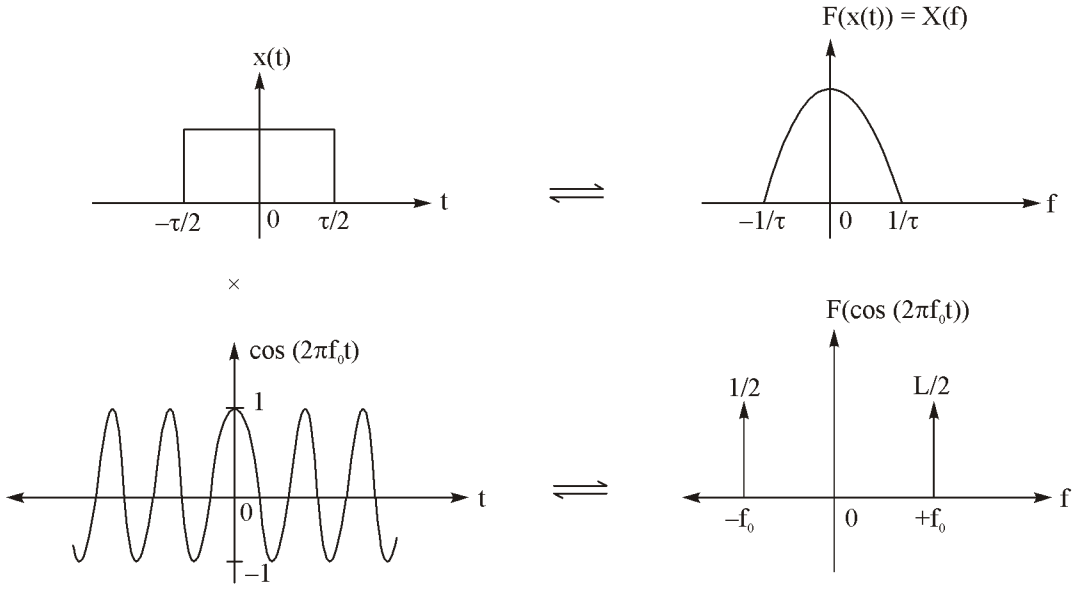
$$x(t)e^{j2\pi f_0 t} + x(t)e^{-j2\pi f_0 t} \rightleftharpoons X(f-f_0)+X(f+f_0)$$

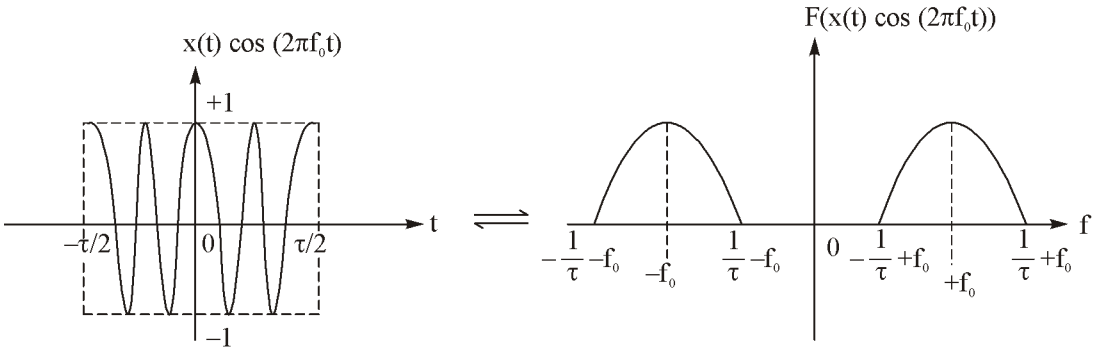
$$x(t)\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right] \rightleftharpoons X(f-f_0)+X(f+f_0)$$

$$x(t)\left[2\cos(2\pi f_0 t)\right] \rightleftharpoons X(f-f_0)+X(f+f_0)$$

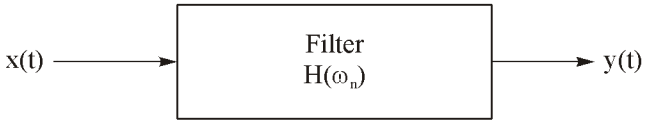
$$x(t)\cos(2\pi f_0 t) \rightleftharpoons \frac{1}{2}\left[X(f-f_0)+X(f+f_0)\right]$$

It means that when the signal x(t) is multiplied with cosine signal, then fourier transform of x(t) is shifted by $+f_0$ and $-f_0$.





10. RESPONSE OF A LINEAR SYSTEM



$$H(\omega_n) = |H(\omega_n)| e^{-j\theta(\omega_n)}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n e^{j2\pi nt/T_0}$$ (in terms of fourier series)

then $$\boxed{y(t) = \sum_{n=-\infty}^{\infty} H(\omega_n) x_n e^{j2\pi nt/T_0}}$$

11. NORMALIZED POWER IN FOURIER SERIES

(i) Normalized power in trigonometric fourier series

$$S = a_0^2 + \sum_{n=1}^{\infty} \frac{a_n^2}{2} + \sum_{n=1}^{\infty} \frac{b_n^2}{2}$$

(ii) Normalized power in polar fourier series

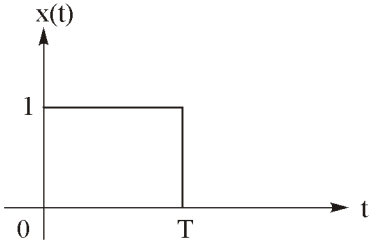
$$S = C_0^2 + \sum_{n=1}^{\infty} \frac{C_n^2}{2}$$

(iii) Normalized power in complex / exponential fourier series

$$S = \sum_{n=-\infty}^{\infty} D_n D_n^*$$

Example: Find the fourier transform of rectangular pulse.

Solution: Rectangular pulse is defined as



$$x(t) = \begin{cases} 1 & ; \quad 0 < t < T \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Fourier transform of $x(t) = x(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

$$= \int_{0}^{T} e^{-j2\pi ft}dt = \left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_0^T$$

$$= -\frac{1}{j2\pi f}\left[e^{-j2\pi fT} - 1\right]$$

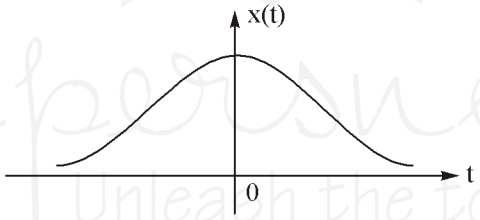
$$= -\frac{1}{j2\pi f}\frac{\left[e^{-j2\pi fT/2} - e^{j2\pi fT/2}\right]}{e^{j2\pi fT/2}}$$

$$= Te^{-j\omega T/2}\sin c\left(\frac{\omega T}{2\pi}\right)$$

Example: Find the fourier transform of Gaussian pulse.

Solution: A Gaussian pulse is defined as

$$x(t) = e^{-\pi t^2}$$



Fourier transform of $x(t) = F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

$$F\left[x(t)\right] = \int_{-\infty}^{\infty} e^{-\pi t^2}e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-\left[\pi t^2 + j\omega t\right]}dt$$

Put $\pi t^2 + j\omega t = \left(\sqrt{\pi}t + \frac{j\omega}{2\sqrt{\pi}}\right)^2 + \frac{\omega^2}{4\pi}$

$$F\left[x(t)\right] = \int_{-\infty}^{\infty} e^{-\left(\sqrt{\pi}t + \frac{j\omega}{2\sqrt{\pi}}\right)^2} e^{-\omega^2/4\pi}dt$$

$$= e^{-\pi f^2}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\pi}t + \frac{j\omega}{2\sqrt{\pi}}\right)^2} dt$$

Assuming $\sqrt{\pi}\,t + \dfrac{j\omega}{2\sqrt{\pi}} = y$

$$\sqrt{\pi}\,dt = dy$$

$$dt = \frac{dy}{\sqrt{\pi}}$$

$$F\left[x(t)\right] = e^{-\pi f^2}\int_{-\infty}^{\infty} e^{-y^2}\,dy/\sqrt{\pi} = \frac{e^{-\pi f^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}dy$$

$$= \frac{e^{-\pi f^2}}{\sqrt{\pi}}\,2\int_{0}^{\infty} e^{-y^2}dy \qquad \text{(because } e^{-y^2} \text{ is an even function)}$$

$$= \frac{2}{\sqrt{\pi}}e^{-\pi f^2}\int_{0}^{\infty} e^{-y^2}dy$$

$$= \frac{\cancel{2}}{\cancel{\sqrt{\pi}}}e^{-\pi f^2}\cdot\frac{\cancel{\sqrt{\pi}}}{\cancel{2}} \qquad \left[\because \int_{0}^{\infty} e^{-y^2}dy = \frac{\sqrt{\pi}}{2}\right]$$

$$F\left[e^{-\pi t^2}\right] = e^{-\pi f^2}$$

$$e^{-\pi t^2} \rightleftharpoons e^{-\pi f^2}$$

12. SOME IMPORTANT PROPERTIES OF FOURIER TRANSFORM

(i) Time Scaling Property

If $x(t) \rightleftharpoons X(f)$

Then $x(at) \rightleftharpoons \dfrac{1}{|a|}X\left(\dfrac{f}{a}\right)$

where, a = any real constant.

(ii) Time Shifting Property

If $x(t) \rightleftharpoons X(f)$

Then $x(t-b) \rightleftharpoons X(f)e^{-j2\pi fb}$

(iii) Time Differentiation Property

If $x(t) \rightleftharpoons X(f)$

Then $\dfrac{d}{dt}x(t) \rightleftharpoons (j2\pi f)X(f)$

(iv) Area Under the Curve

If $x(t) \rightleftharpoons X(f)$

then $\int_{-\infty}^{\infty} x(t)dt = X(0)$ = Area under the curve x(t)

and $\int_{-\infty}^{\infty} X(f)df = x(0)$ = Area under the curve X(f)

13. CONVOLUTION

It is a mathematical operation which is used to express the input / output relationship in a linear time invariant system.

It is represented by $*$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

(a) Convolution Theorem in Time Domain

If $x_1(t) \rightleftharpoons X_1(f)$

$x_2(t) \rightleftharpoons X_2(f)$

then $x_1(t) * x_2(t) \rightleftharpoons X_1(f)X_2(f)$

(b) Convolution Theorem in Frequency Domain

If $x_1(t) \rightleftharpoons X_1(f)$

$x_2(t) \rightleftharpoons X_2(f)$

then $x_1(t)x_2(t) \rightleftharpoons X_1(f) * X_2(f)$

14. ENERGY SIGNAL

Energy signal has finite energy and zero average power.

x(t) is non periodic signal or time limited signal, then energy is expressed as

$$E = \int_{-\infty}^{\infty} x^2(t)dt$$

If $x(t) \rightleftharpoons X(f)$, then

$$E = \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$|X(f)|^2 = \frac{E}{\Delta f} = \text{Energy spectral density} = \text{Energy density spectrum.}$$

$$= \text{energy per unit B.W.} = \psi(f)$$

15. POWER SIGNAL

A power signal has finite power and infinite energy.

All periodic signal is power signal.

$$0 < P < \infty,\ E = \infty$$

where, P = average power and E is energy of signal.

If x(t) is periodic signal, then

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt$$

If complex fourier series of $x(t) = \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_0 t)$

then power (in terms of complex fourier series coefficient.

$$= \sum_{n=-\infty}^{\infty} |C_n|^2$$

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PRACTICE SHEET

OBJECTIVE QUESTIONS

1. Let x(t) be a real signal with the Fourier transform X(f). Let X′(f) denote the complex conjugate of X(f). Then

(a) $X(-f) = X'(f)$ (b) $X(-f) = X(f)$

(c) $X(-f) = -X(f)$ (d) $X(-f) = -X'(f)$

2. Let the transfer function of a network be $H(f) = |H(f)|\, e^{j\theta(f)}$. If a signal x(t) is applied to such a network, the output Y(t) is given by

(a) 2 x(t) (b) x (t–2)

(c) 2x (t–2) (d) 2x(t–4π)

3. Let $m(t) = \cos[(2\pi \times 10^3)t]$ be the message signal and $c(t) = 15 \cos [(2\pi \times 10^6)t]$ be the carrier. c(t) and m(t) are to generate an AM signal. The modulation index of the generated AM signal is 0.5. The ratio of the upper sideband power to the carrier power

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{8}$ (d) $\frac{1}{16}$

4. The trigonometric Fourier series of a periodic time function can have only

(a) cosine terms

(b) sine terms

(c) cosine and sine terms

(d) DC and cosine terms

5. Which of the following cannot be the Fourier series expansion of a periodic signal?

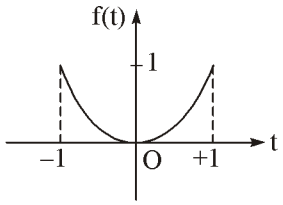
(a) x(t) = 2 cost + 3 cos 3t

(b) x(t) = 2 cos πt + 7 cost

(c) x(t) = cost + 0.5

(d) x(t) = 2cos 1.5πt + sin 3.5 πt

6. The Fourier transform of given figure with ω = 1 rad/sec



(a) $\left(\cos\frac{\pi}{180} + \sin\frac{\pi}{180}\right)$

(b) $\left(4\cos\frac{\pi}{180} + \sin\frac{\pi}{180}\right)$

(c) $\left(2\cos\frac{\pi}{180} + 4\sin\frac{\pi}{180}\right)$

(d) $\left(2\sin\frac{\pi}{180} + 4\cos\frac{\pi}{180}\right)$

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ANSWERS AND EXPLANATIONS

1. Ans. (a)

$x(t) \leftrightarrow X(f)$

we know that

$x(-t) \leftrightarrow X(-f)$

and $x^*(t) \leftrightarrow X^*(-f)$

$\because$ x(t) is a real signal

$\therefore \quad x(t) = X^*(t)$

$\therefore \quad x(f) = X^*(-f)$

or $\quad x(-f) = X^*(f)$

2. Ans. (c)

Given $H(f) = 2e^{-j4\pi f}$

and $x(t) \leftrightarrow x(f)$

$\because$ o/p $Y(t) \leftrightarrow x(t) * h(t)$

Taking Fourier transform we get

$Y(f) = X(f)\ H(f)$

(according to convolution property of F.T.)

$$Y(f) = X(f)2e^{-J\pi\times 4f}$$

$$Y(f) = 2X(f)e^{-J2\pi f\times 2}$$

taking in verse F.T. we get

$$Y(t) = 2F\cdot T^{-1}\left[X(f)e^{-j\pi f\times 2}\right] \qquad ...(i)$$

we know that

Time shifting property

$$x(t-t_0) \leftrightarrow x(f)e^{-j2\pi ft_0} \qquad ...(ii)$$

so from (1) and (2) we have

$Y(t) = 2x(t - 2)$

3. Ans. (d)

$$\frac{\text{Upper side band power}}{\text{Carrier power}} = \frac{\mu^2}{4} = \frac{(0.5)^2}{4} = \frac{1}{16}$$

4. Ans. (d)

The trigonometric Fourier series of periodic time function can have DC and cosine terms.

5. Ans. (b)

$x(t) = (2 \cos \pi t + 7 \cot t)$ does not satisfy the Drichlet condition, therefore, it can't be expanded as Fourier series.

6. Ans. (d)

Signal is

$$f(t) = t^2; -1 \le t \le +1$$

Fourier transfer of signal is,

$$F[f(t)] = \int_{-1}^{+1} t^2 e^{-j\omega t}dt$$

$$= \frac{1}{j\omega}\left(e^{j\omega} - e^{-j\omega}\right) + \frac{2}{\omega^2}\left(e^{-j\omega} + e^{j\omega}\right)$$

$$+ \frac{2}{j\omega^3}\left(e^{-j\omega} - e^{j\omega}\right)$$

if $\omega = 1$ rad/sec.

$$F(f(t)) = 2\left(e^{-j} + e^{j}\right) + j\left(e^{j} - e^{-j}\right)$$

$$= 4\cos\frac{\pi}{180^o} + 2\sin\frac{\pi}{180^o}$$

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