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CBSE – XIIth

Maths

Central Board of Secondary Education (CBSE)

Quick Revision Notes + Sample Questions

SIERRA INNOVATIONS

INDEX

01

Relation and Function

Relation :

If A and B are two non-empty sets, then a relation R from A to B is a subset of A × B.

If $R \subseteq A \times B$ and $(a, b) \in R$, then we say that a is related to b by the relation R, written as aRb.

If $R \subseteq A \times A$, then we simply say R is a relation on A.

Representation of a Relation :

(i) **Roster form** In this form, we represent the relation by the set of all ordered pairs belongs to R.

e.g. Let R is a relation from set

A = { –3, – 2, – 1,1, 2, 3} to set

B = {1, 4, 9,10}, defined by $aRb \Leftrightarrow a^2 = b$,

Then, $(-3)^2 = 9, (-2)^2 = 4, (-1)^2 = 1, (2)^2 = 4, (3)^2 = 9$.

Then, in roster form, R can be written as

R = {(–1,1), (– 2, 4), (1,1),(2, 4),(– 3, 9),(3, 9)}

(ii) **Set-builder form** In this form, we represent the relation R from set A to set B as

R = {(a, b) : a ∈ A, b∈B and the rule which relate the elements of A and B}

e.g. Let R is a relation from set A = {1, 2, 4, 5} to set

$$B = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}\right\} \text{ such that}$$

$$R = \left\{(1,1), \left(2, \frac{1}{2}\right)\left(4, \frac{1}{4}\right)\left(5, \frac{1}{5}\right)\right\}$$

Then, in set-builder form, R can be written as

$$R = \left\{(a, b) : a \in A, b \in B \text{ and } b = \frac{1}{a}\right\}$$

Note :

We cannot write every relation from set A to set B in set-builder form.

Domain, Codomain and Range of a Relation:

Let R be a relation from a non-empty set A to a non-empty set B. Then, set of all first components or coordinates of the ordered pairs belonging to R is called the **domain** of R, while the set of all second components or coordinates of the ordered pairs belonging to R is called the **range** of R Also, the set B is called the **codomain** of relation R.

Thus, domain of R = {a : (a, b)∈R} and range of R = { b : (a, b)∈R}

Types of Relations:

(i) **mpty or Void Relation** As $\phi \subset A \times A$, for any set A, so ϕ is a relation on A, called the empty or void relation.

(ii) **Universal Relation** Since, $A \times A \subseteq A \times A$, so A × A is a relation on A, called the universal relation.

(iii) **Identity Relation** The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.

(iv) **Reflexive Relation** A relation R on a set A is said to be reflexive relation, if every element of A is related to itself.

Thus, $(a, a) \in R, \forall a \in A \Rightarrow R$ is reflexive.

(v) **Symmetric Relation** A relation R on a set A is said to be symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$

i.e. $a R b \Rightarrow bRa, \forall a, b \in A$

(vi) **Transitive Relation** A relation R on a set A is said to be transitive relation, iff (a, b)∈R and (b, c)∈ R $\Rightarrow (a, c) \in R, \forall a, b, c \in A$

Equivalence Relation :

A relation R on a set A is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on A.

Equivalence Classes:

Let R be an equivalence relation on A (≠ ϕ). Let a ∈ A Then, the equivalence class of a denoted by [a] or (a) is defined as the set of all those points of A which are related to a under the relation R.

Inverse Relation:

If A and B are two non-empty sets and R be a relation from A to B, then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Composition of Relation :

Let R and S be two relations from sets A to B and B to C respectively, then we can define relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\, b \in B$ such that (a, b)∈R and (b, c)∈S.

This relation SoR is called the composition of R and S.

(i) RoS ≠ SoR (ii) $(SoR)^{-1} = R^{-1}oS^{-1}$ known as **reversal rule**.

Important Results on Relation:

(i) If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

(ii) The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

(iii) If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

(vi) Let A and B be two non-empty finite sets consisting of m and n elements, respectively. Then, A × B consists of mn ordered pairs. So, the total number of relations from A to B is 2^{nm}.

(v) If a set A has n elements, then number of reflexive relations from A to A is 2^{n^2-n}.

Function

Let A and B be two non-empty sets, then a function f from set A to set B is a rule which associates each element of A to a unique element of B.

It is represented as $f : A \to B$ or $A \xrightarrow{f} B$ and function is also called the mapping.

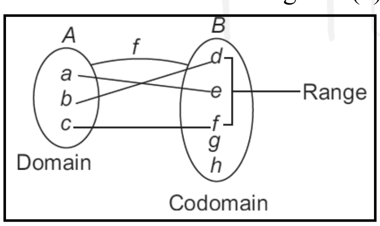
Domain, Co–domain and Range of a Function :

If $f : A \to B$ is a function from A to B, then

(i) the set A is called the domain of f(x).

(ii) the set B is called the codomain of f(x).

(iii) the subset of B containing only the images of elements of A is called the range of f(x).



Characteristics of a Function $f : A \to B$:

(i) For each element x Î A, there is unique element y ÎB.

(ii) The element y ÎBis called the image of x under the function f.

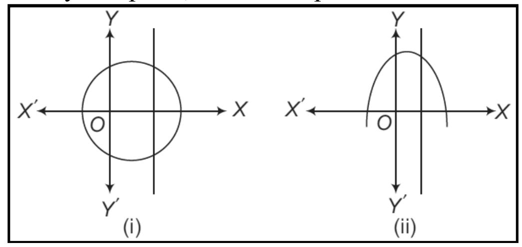
Also, y is called the value of function f at x i.e. f(x) = y.

(iii) $f : A \to$ Bis not a function, if there is an element in Awhich has more than one image in B. But more than one element of Amay be associated to the same element of B.

(iv) $f : A \to B$ is not a function, if an element in A does not have an image in B.

Identification of a Function from its Graph:

Let us draw a vertical line parallel toY-axis, such that it intersects the graph of the given expression. If it intersects the graph at more than one point, then the expression is a relation else, if it intersects at only one point, then the expression is a function.



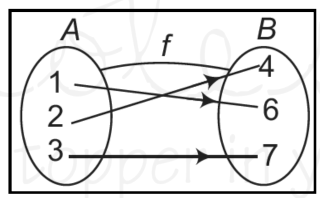
In figure (i), the vertical parallel line intersects the curve at two points, thus the expression is a relation whereas in figure (ii), the vertical parallel line intersects the curve at one point. So, the expression is a function.

Types of Functions :

1. One-One (or Injective) Function :

A mapping $f : A \to B$ is a called one-one (or injective) function, if different elements in A have different images in B, such a mapping is known as one-one or injective function.

Methods to Test One-One

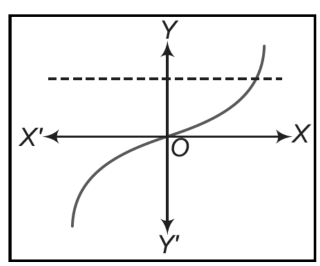


(i) Analytically If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or quivalently $x_1 \neq x_2$

$\Rightarrow f(x_1) \neq f(x_2),\ \forall\ x_1 = x_2 \in A$,

then the function is one-one.

(ii) **Graphically** If every line parallel to X-axis cuts the graph of the function atmost at one point, then the function is one-one.



(iii) Monotonically If the function is increasing or decreasing in whole domain, then the function is one-one.

Number of One-One Functions

Let A and B are finite sets having m and n elements respectively, then the number of one-one functions

from A to B is $\begin{cases} {}^nP_m, n \geq m \\ 0, n < m \end{cases}$

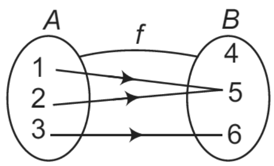
$= \begin{cases} n(n-1)(n-2)...(n-(m-1)), n \geq m \\ 0, \quad n < m \end{cases}$

2. ***Many-One Function :***

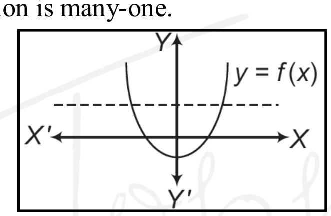
A function f : A → B is called many-one function, if two or more than two different elements in A have the same image in B.

Method to Test Many-One :

(i) ***Analytically*** If $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$ for some x_1, $x_2 \in A$, then the function is many-one.



(ii) ***Graphically*** If any line parallel to X-axis cuts the graph of the function atleast two points, then the function is many-one.



(iii) ***Monotonically*** If the function is neither strictly increasing nor strictly decreasing, then the function is many-one.

Number of Many-One Function :

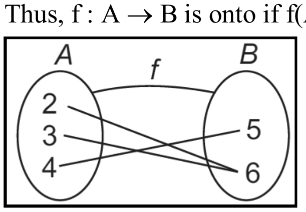
Let A and B are finite sets having m and n elements respectively, then the number of many-one function from A to B is

= Total number of functions − Number of one-one functions

$\begin{cases} n^m - {}^nP_m, \text{if } n \geq m \\ n^m, \text{if } n < m \end{cases}$

3. ***Onto (or Surjective) Function :***

If the function f : A → B is such that each element in B (co–domain) is the image of at least one element of A, then we say that f is a function of A onto B. Thus, f : A → B is onto if f(A) = B.



i.e. Range = Co–domain

Note Every polynomial function f : R→ R of odd degree is onto.

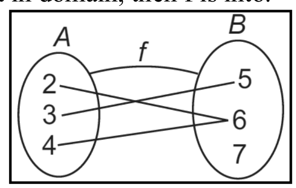
Number of Onto** (or Surjective) **Functions

Let A and B are finite sets having m and n elements respectively, then number of onto (or surjective) functions from A to B is

$= \begin{cases} n^m - {}^nC_1(n-1)^m + C_2(n-2)^m - {}^nC_3(n-3)^m + ..., & n < m \\ n! & n = m \\ 0, & n > m \end{cases}$

4. ***Into Function :***

If f : A → B is such that there exists atleast one element in codomain which is not the image of any element in domain, then f is into.



Thus, f : A → B, is into iff f(A) ⊂ B

i.e. Range ⊂ Codomain

Number of Into Function :

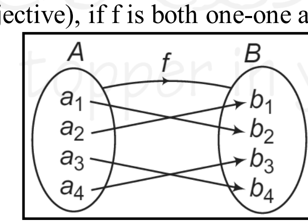
Let A and B be finite sets having m and n elements respectively, then

number of into functions from A to B is

$= \begin{cases} {}^nC_1(n-1)^m + C_2(n-2)^m - {}^nC_3(n-3)^m + ..., & n \leq m \\ n^m, & n > m \end{cases}$

5. ***One-One and Onto Function (or Bijective) :***

A function f : A→ B is said to be one-one and onto (or bijective), if f is both one-one and onto.



Number of Bijective Functions :

Let A and B are finite sets having m and n elements respectively, then number of onto functions from A

to B is $\begin{cases} n!, \text{if } n = m \\ 0, \text{if } n > m \text{ or } n > m \end{cases}$

Equal Functions :

Two functions f and g are said to be equal if f

(i) domain of f = domain of g.

(ii) codomain of f = codomain of g.

(iii) f(x) = g(x) for every x belonging to their common domain and then we write f = g.

Real Valued and Real Functions :

A function f : A→ B is called a real valued function, if B£ R and it is called a real function if, A£ R and B£ R.

2. *Domain of Real Functions :*

The domain of the real function f(x) is the set of all those real numbers for which the expression for f (x) or the formula for f(x) assumes real values only.

Range of Real Functions:

The range of a real function of a real variable is the set of all real values taken by f (x) at points of its domain.

Working Rule for Finding Range of Real Functions :

Let $y = f(x)$ be a real function, then for finding the range we may use the following steps

Step I : Find the domain of the function $y = f(x)$.

Step I : Transform the equation $y = f(x)$ as $x = g(y)$. i.e. convert x in terms of y.

Step III : Find the values of y from $x = g(y)$ such that the values of x are real and lying in the domain of f.

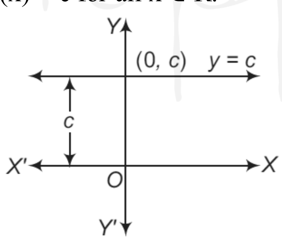
Step IV: The set of values of y obtained in step III be the range of function f.

Standard Real Functions and their Graphs :

1. *Constant Function :*

Let c be a fixed real number. The function which associates each real number x to this fixed number c, is called a constant function.

i.e. $y = f(x) = c$ for all $x \in R$.



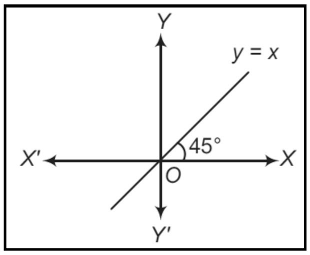
Domain of f(x) = R and Range of f(x) = {c}.

2. *Identity Function :*

The function which associates each real number x to the same number x, is called the identity function.

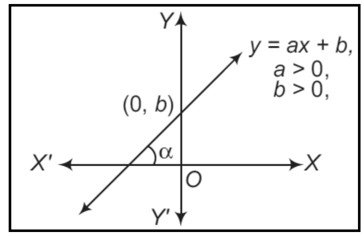
i.e. y = f(x) = x,"x ÎR.

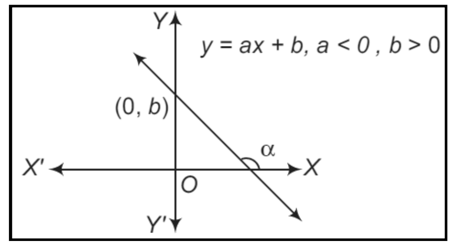
Domain of f(x) = R and Range of f(x) = R



3. *Linear Function :*

If a and b are fixed real numbers, then the linear function is defined as $y = f(x) = ax + b$. The graph of a linear function is given in the following diagram, which is a straight line with slope tanα.





Domain of f(x) = R and Range of f(x) = R.

4. *Quadratic Function :*

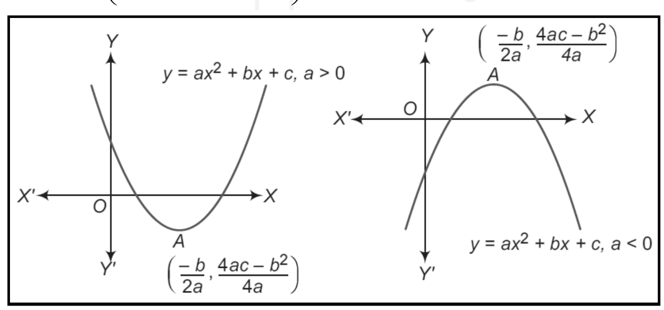
If a, b and c are fixed real numbers, then the quadratic function is expressed as

$y = f(x) = ax^2 + bx + c, a \neq 0$

$$\Rightarrow y = a\left(-\frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

which represents a downward parabola, if $a < 0$ and upward parabola, if $a > 0$ and vertex of this parabola is at

$$\left(-\frac{b}{2a}, +\frac{4ac - b^2}{4a}\right)$$



Domain of f(x) = R

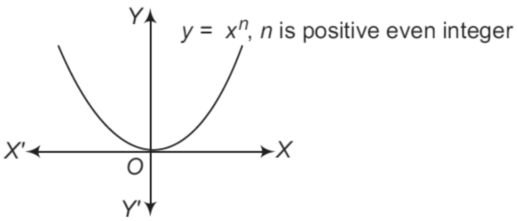
Range of f(x) is $\left(-\infty, \frac{4ac - b^2}{4a}\right]$, if $a < 0$ and $\left[\frac{4ac - b^2}{4a}, \infty\right)$, if $a > 0$.

5. *Power Function :*

The power function is given by $y = f(x) = x^n$, $n \in I$, $n \neq 1, 0$.

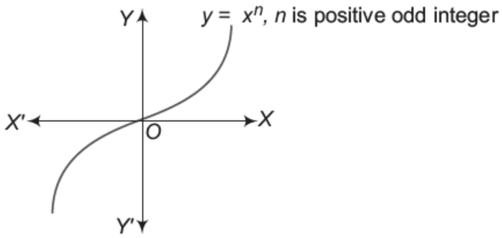
The domain and range of $y = f(x)$, is depend on n.

(a) If n is positive even integer, i.e. $f(x) = x^2, x^4, \ldots$



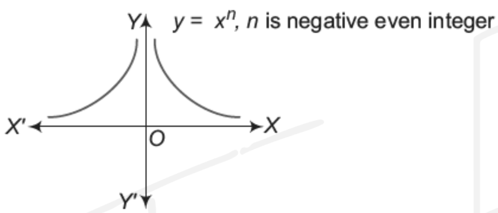
Domain of f(x) = R and Range of f(x) = $[0, \infty)$

(b) If n is positive odd integer, i.e. $f(x) = x^3, x^5, \ldots$



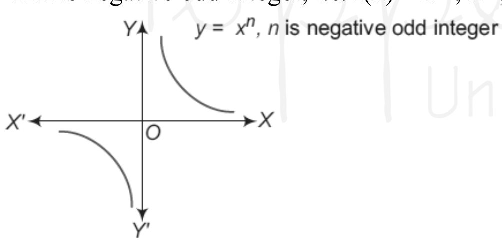
Domain of f(x) = R and Range of f(x) = R

(c) If n is negative even integer, i.e. $f(x) = x^{-2}, x^{-4}, \ldots$



Domain of f(x) = R − {0} and Range of f(x) = $(0, \infty)$

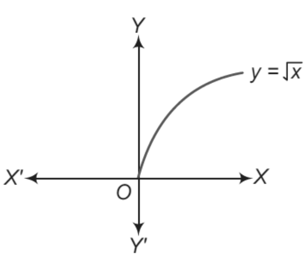
(d) If n is negative odd integer, i.e. $f(x) = x^{-1}, x^{-3}, \ldots$



Domain of f(x) = R − { 0} and Range of f(x) = R − {0}

6. ***Square Root Function :***

Square root function is defined by $y = f(x) = \sqrt{x}$, $x \geq 0$.

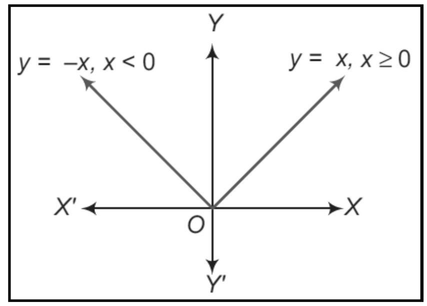


Domain of f(x) = $[0, \infty)$ and Range of f(x) = $[0, \infty)$

7. ***Modulus (or Absolute Value) Function :***

Modulus function is given by y = f(x) =|x|, where |x| denotes the absolute value of x,

i.e. $|x| = \begin{cases} x, \text{if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$

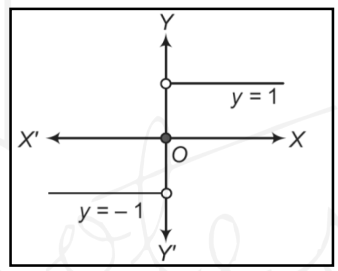


Domain of f(x) = R and Range of f(x) = $[0, \infty)$.

8. ***Signum Function :***

Signum function is defined as follows

$y = f(x) \begin{cases} \frac{|x|}{x}, \text{if } x \neq 0 \\ 0, \quad \text{if } x = 0 \end{cases}$ or $\begin{cases} \frac{x}{|x|}, \text{if } x \neq 0 \\ 0, \quad \text{if } x = 0 \end{cases}$



Symbolically, signum function is denoted by sgn (x).

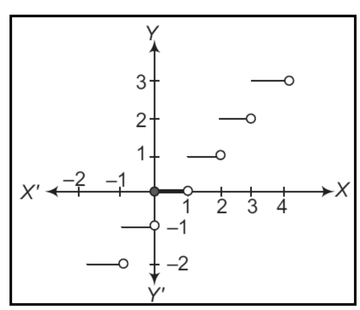
Thus, $y = f(x) = \text{sgn}(x)$

where, $\text{sgn}(x) = \begin{cases} -1, \text{if } x < 0 \\ 0, \text{if } x = 0 \\ 1, \text{if } x > 0 \end{cases}$

Domain of sgn (x) = R and Range of sgn (x) = {−1, 0,1}

9. ***Greatest Integer Function/Step Function/ Floor Function :***

The greatest integer function is defined as y = f(x) = [x]



where, [x] represents the greatest integer less than or equal to x. In general, if $n \le x < n + 1$ for any integer n, [x] = n.

Thus, [2.304] = 2, [4] = 4 and [– 8.05] = – 9

X	[x]
0 < x < 1	0
1 < x < 2	1
–1 < x < 0	–1
– 2 ≤ x < – 1	–2
...	...

Domain of f(x) = R and Range of f(x) = I, the set of integers.

Properties of Greatest Integer Function

(i) $[x + n] = n + [x], n \in I$

(ii) $[-x] = -[x], x \in I$

(iii) $[-x] = -[x] - 1, x \notin I$

(iv) $[x] \le n \Rightarrow x \ge n, n \in I$

(v) $[x] \le n \Rightarrow x \ge n + 1, n \in I$

(vi) $[x] < n \Rightarrow x < n + 1, n \in I$

(vii) $[x] < n \Rightarrow x < n, n \in I$

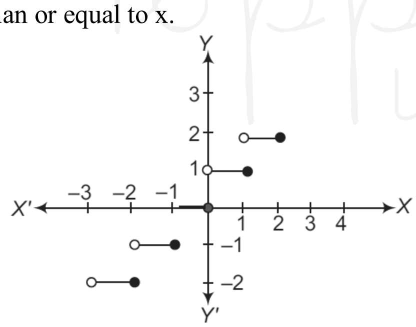
(viii) $[x + y] = [x] + [y + x - [x]]$ for all $x, y \in R$

(ix) $[x + y] \ge [x] + [y]$

(x) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+...+\left[x+\frac{n-1}{n}\right]=[nx], n \in N$

10. *Least Integer Function/Ceiling Function/ Smallest Function :*

The least integer function is defined as y = f(x) = (x), where (x) represents the least integer greater than or equal to x.



Thus, (3.578) = 4, (0.87) = 1, (4) = 4, (− 8.239) = − 8, (− 0.7) = 0

In general, if n is an integer and x is any real number such that n < x £ n + 1, then (x) = n + 1

∴ f(x) = (x) = [x] + 1

x	[x]
–1 < x ≤ 0	0
0 < x ≤ 1	1
1 < x ≤ 2	2
2 < x ≤ 3	3
– 2 ≤ x < – 1	–1
...	...

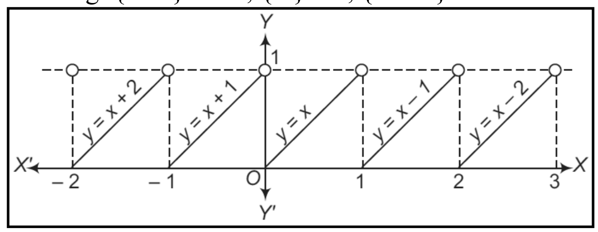
Domain of f = R and Range of f = I

11. *Fractional Part Function :*

It is defined as f(x) = {x}, where { x} represents the fractional part of x,

i.e., if x = n + f, where n ÎI and 0 ≤ f < 1, then { x} = f

e.g. { 0. 7} = 0.7, { 3} = 0, { − 3.6} = 0.4



Properties of Fractional Part Function :

(i) {x} = x − [x]

(ii) {x} = x, if 0 < x < 1

(iii) {x} = 0, if $x \in I$

(iv) {−x} = 1 − {x}, if $x \notin I$

12. *Exponential Function :*

Exponential function is given by $y = f(x) = a^x$, where a > 0, a ≠ 1.

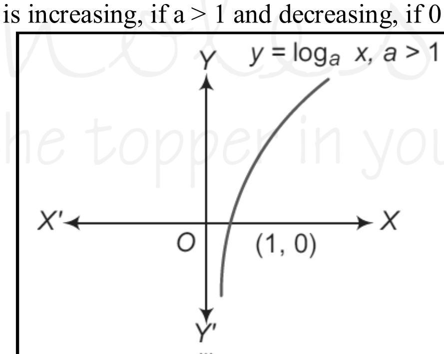
The graph of the exponential function is as shown, which is increasing, if a > 1 and decreasing, if 0 < a < 1.

Domain of f(x) = R and Range of f(x) = (0, ∞)

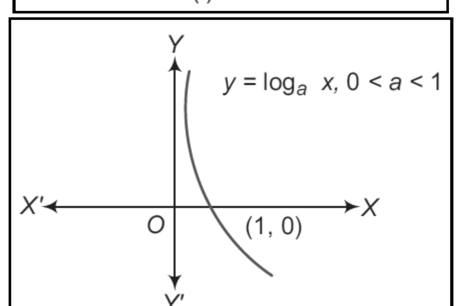
13. *Logarithmic Function :*

A logarithmic function may be given by y = f(x) = $\log_a x$, where a > 0, a ≠ 1 and x > 0.

The graph of the function is as shown below, which is increasing, if a > 1 and decreasing, if 0 < a < 1.



(i)



(ii)

Domain of f(x) = (0, ∞) and Range of f(x) = R

Operations on Real Functions :

Let f : A → B and g : A → B be two real functions, then

(i) **Addition** The addition f + g is defined as
$f + g : A \to B$ such that $(f + g)(x) = f(x) + g(x)$.

(ii) Difference The difference f − g is defined as $f - g : A \to B$ such that $(f - g)(x) = f(x) - g(x)$.

(iii) Product The product f g is defined as
$fg : A \to B$ such that $(fg)(x) = f(x)\, g(x)$.
Clearly, f ± g and fg are defined only, if f and g have the same domain. In case, the domain of f and g are different, then domain of f + g or fg = domain of f ∩ domain of g.

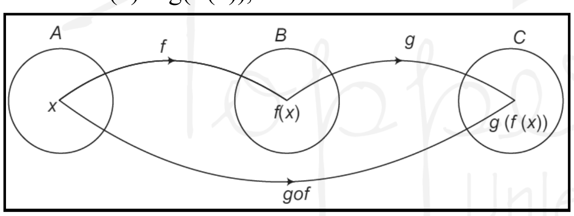
(iv) **Multiplication by a Number** (or a Scalar) The function cf, where c is a real number is defined as $cf : A \to B$, such that $(cf)(x) = cf(x)$.

(v) **Quotient** The quotient $\frac{f}{g}$ is defined as

$\frac{f}{g} : A \to B$ such that $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

Composition of Two Functions :

Let $f : A \to B$ and $g : B \to C$ be two functions. Then, we define
Go f : $A \to C$, such that
Go f (x) = g(f(x)), $\forall\, x \in A$



Important Points to be Remembered :

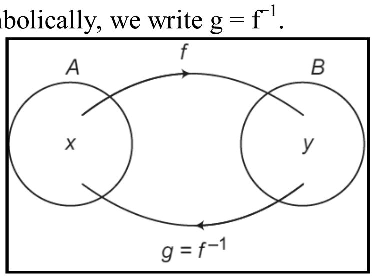
(i) If f and gare injective, then fogand gof are injective.

(ii) If f and gare surjective, then fogand gof are surjective.

(iii) If f and gare bijective, then fog and gof are bijective.

Inverse of a Function :

Let $f : A \longrightarrow B$ is a bijective function, i.e. it is one-one and onto function. Then, we can define a function $g : B \longrightarrow A$, such that $f(x) = y \Rightarrow g(y) = x$, which is called inverse of f and vice-versa. Symbolically, we write $g = f^{-1}$.



A function whose inverse exists, is called an invertible function or inversible.
(i) Domain (f^{-1}) = Range (f)
(ii) Range (f^{-1}) = Domain (f)
(iii) If f(x) = y, then $f^{-1}(y) = x$ and vice-verse.

Periodic Functions :

A function f(x) is said to be a periodic function of x, if there exists a real number T > 0, such that f(T + x) = f(x)," x ∈Dom(f).
The smallest positive real number T, satisfying the above condition is known as the period or the fundamental period of f(x).

Testing the Periodicity of a Function :

(i) Put f(T + x) = f(x) and solve this equation to find the positive values of T independent of x.

(ii) If no positive value of T independent of x is obtained, then f(x) is a non-periodic function.

(iii) If positive values of T which is independent of x are obtained, then f(x) is a periodic function and the least positive value of T is the period of the function f(x).

Important Points to be Remembered :

(i) Constant function is periodic with no fundamental period.

(ii) If f (x) is periodic with period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ are also periodic with same period T.

(iii) If f (x) is periodic with period T_1 and g(x) is periodic with period T_2, then f(x) + g(x) is periodic with period equal to

(a) LCM of $\{T_1, T_2\}$, if there is no positive k, such that f (k + x) = g(x) and g(k + x) =f (x).

(b) 1/2 LCM of $\{T_1, T_2\}$, if there exist a positive number k such that f (k + x) = g(x) and g(k + x) =f (x)

(iv) If f (x) is periodic with periodic T, then kf (ax + b) is periodic with period $\frac{T}{|a|}$, where $a, b, k \in R$ and $a, k \neq 0$.

(v) If f (x) is a periodic function with period T and g(x) is any function, such that range of f ⊆ domain of g, then g of is also periodic with period T.

Even and Odd Functions :

Even Function A real function f(x) is an even function, if f(−x) = f(x).
Odd Function A real function f(x) is an odd function, if f(−x) = − f(x).

Properties of Even and Odd Functions :

(i) Even function ± Even function = Even function.
(ii) Odd function ± Odd function = Odd function.
(iii) Even function × Odd function = Odd function.
(iv) Even function × Even function = Even function.
(v) Odd function × Odd function = Even function.
(vi) go f or fog is even, if both f and g are even or if f is odd and g is even or if f is even and g is odd.
(vii) go f or fog is odd, if both of f and g are odd.

(viii) If f(x) is an even function, then $\frac{d}{dx}f(x)$ or $\int f(x)dx$ is an odd function and if f(x) is an odd function, then $\frac{d}{dx}f(x)$ or $\int f(x)dx$ is an even function.

(ix) The graph of an even function is symmetrical about Y -axis.

(x) The graph of an odd function is symmetrical about origin or symmetrical in opposite quadrants.

(xi) An every function can never be one-one, however an odd function may or may not be one-one.

Binary Operations :

Let S be a non-empty set. A function * from S × S to S is called a binary operation on S i.e. * : S ×S → S is a binary operation on set S.

Note Generally binary operations are represented by the symbols *, ⊕, ... etc., instead of letters figure etc.

Closure Property :

An operation * on a non-empty set S is said to satisfy the closure property, if

a∈S,b∈S ⇒a* b∈S, ∀ a, b∈S

Also, in this case we say that S is closed under *.

An operation * on a non-empty set S, satisfying the closure property is known as a binary operation.

Some Particular Cases :

(i) Addition is a binary operation on each one of the sets N, Z, Q, R and C, i.e. on the set of natural numbers, integers, rationals, real and complex numbers, respectively. While addition on the set S of all irrationals is not a binary operation.

(ii) Multiplication is a binary operation on each one of the sets N, Z, Q, R and C, i.e. on the set of natural numbers, integers, rationals, real and complex numbers, respectively. While multiplication on the set S of all irrationals is not a binary operation.

(iii) Subtraction is a binary operation on each one of the sets Z, Q, R and C, i.e. on the set of integers, rationals, real and complex numbers, respectively. While subtraction on the set of natural numbers is not a binary operation.

(iv) Let S be a non-empty set and P (S) be its power set. Then, the union, intersection and difference of sets, on P (S) is a binary operation.

(v) Division is not a binary operation on any of the sets N, Z, Q, R and C. However, it is a binary operation on the sets of all non-zero rational (real or complex) numbers.

(vi) Exponential operation (a, b) → a^b is a binary operation on set N of natural numbers while it is not a binary operation on set Z of integers.

Properties of Binary Operations :

(i) Commutative Property A binary operation * on a non-empty set S is said to be commutative or abelian, if

a * b = b * a, ∀ a, b ∈S.

Addition and multiplication are commutative binary operations on Z but subtraction is not a commutative binary operation, since $2-3 \neq 3-2$.

Union and intersection are commutative binary operations on the power set P (S) of S. But difference of sets is not a commutative binary operation on P (S).

(ii) Associative Property **A** ry operation * on a non-empty set S is said to be associative, if (a * b) * c = a * (b * c), ∀ a, b, c ∈S.

Let R be the set of real numbers, then addition and multiplication on R satisfies the associative property.

(iii) Distributive Property Let* and o be two binary operations on a non-empty sets. We say that * is distributed over o, if a* (b o c) = (a* b) o (a* c), ∀b, c ∈S also (called left distributive law) and (b o c) * a = (b * a) o (c * a), ∀ a, b, c ∈S also (called right distributive law).

Let R be the set of all real numbers, then multiplication distributes over addition on R.

Since, a · (b + c) = a · b + a · c, ∀ a, b, c ∈R.

Identity Element :

Let * be a binary operation on a non-empty set S. An element e ∈S, if it exist, such that a * e = e * a = a, ∀ a ∈S, is called an identity elements of S, with respect to *.

For addition on R, zero is the identity element in R.

Since, a + 0 = 0 + a = a, ∀ a∈R

For multiplication on R, 1 is the identity element in R.

Since, a × 1 = 1 × a = a,∀ a∈R

Let P (S) be the power set of a non-empty set S. Then, ϕ is the identity element for union on P (S), as A∪ ϕ = ϕ ∪ A= A,∀ A∈P(S)

Also, S is the identity element for intersection on P (S).

Since, A∩S = A∩S = A, ∀ A∈P (S).

For addition on N the identity element does not exist. But for multiplication on N the identity element is 1.

Inverse of an Element :

Let * be a binary operation on a non-empty set S and let e be the identity element.

Suppose a∈S, we say that a is invertible, if there exists an element b∈S such that a* b = b * a = e

Also, in this case, b is called the inverse of a and we write, $a^{-1} = b$

Addition on N has no identity element and accordingly N has no invertible element.

Multiplication on N has 1 as the identity element and no element other than 1 is invertible.

Important Points to be Remembered :

If S be a finite set containing n elements, then

(i) the total number of binary operations on S in n^{n^2} .

(ii) the total number of commutative binary operations' on S is $n^{n(n+1)/2}$.

Sample Questions CBSE

Types of Relations

MCQ

1. A relation R in set A = {1,2,3} is defined as R = {(1, 1), (1, 2), (2, 2), (3, 3)}. Which of the following ordered pair in R shall be removed to make it an equivalence relation in A? **(Term I, 2021-22)**

(a) (1, 1) (b) (1, 2)

(c) (2, 2) (d) (3, 3)

2. Let the relation R in the set A = {x∈Z : $0 \le x \le 12$}, given by R = {(a, b) : |a – b| is a multiple of 4.} Then [1], **(Term I, 2021-22)**

(a) {1,5,9} (b) {0,1,2,5}

(c) ϕ (d) A

VSA

3. How many reflexive relations are possible in a set A whose n(A) = 3? **(2020-21)**

4. A relation R in S = {1,2,3}is defined as R = {(1, 1), (1, 2), (2, 2), (3, 3)}. Which element(s) of relation R be removed to make R an equivalence relation? **(2020-21)**

5. An equivalence relation R in A divides it into equivalence classes A_1,A_2,A_3. What is the value of $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$. **(2020-21)**

SA–I

6. Let R be the relation in the set Z of integers given by R = {(a,b) : 2 divides a – b}: a + bis “divisible by 2”]is reflexive, symmetric or transitive. Write the equivalence class containing 0, i.e..,[0]. **(2020-21)**

SA–II

7. Check whether the relation R in the set Z of integers defined as R = {(a – b) : a + b is “divisible by 2”} is reflexive, symmetric or transitive. Write the equivalence class containing 0,i.e..,[0] **(2020-21)**

LA–II

8. Given a non–empty set X, define the relation R on P(X) as :

For A, B ∈ P(X), (A, B) ∈ R iff A ⊂ B. Prove that R is reflexive, transitive, and not symmetric. **(2022-23)**

9. Define the relation R in the set N × N as follows:

For (a, b), (c, d) ∈ N × N, (a, b) R (c, d) if f ad = bc.

Prove that R is an equivalence relation in N × N.

(2022-23)

Types of Functions

MCQ

10. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Based on the given information, f is best defined as

(Term I, 2021-22)

(a) Surjective function

(b) Injective function

(c) Bijective function

(d) Function

11. The function f : R → R defined as $f(x) = x^3$ is

(a) One-one but not onto **(Term I, 2021-22)**

(b) Not one-one but onto

(c) Neither one-one nor onto

(d) One-one and onto

VSA

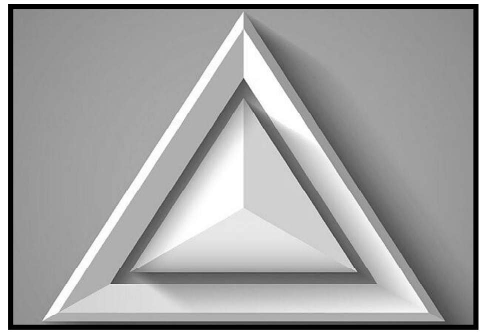
12. Check whether the function f : R →R defined as $f(x) = x^3$ is one-one or not. **(2020-21)**

13. A relation R in the set of real numbers R defined as R = {(a,b) : $\sqrt{a} = b$ } is a function or not. Justify.

(2020-21)

02

Inverse Trignometric Function



Inverse Trigonometric Functions :

If y = f(x) and x = g (y) are two functions such that f (g (y)) = y and g (f(y)) = x, then f and y are said to be inverse of each other, i.e. $g = f^{-1}$. If y = f(x), then $x = f^{-1}(y)$.

Inverse Trigonometric Functions :

As we know that trigonometric functions are not one-one and onto in their natural domain and range, so their inverse do not exist but if we restrict their domain and range, then their inverse may exists.

Domain and Range of Inverse Trigonometric Functions :

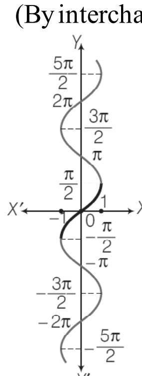
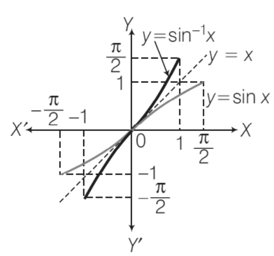
The range of trigonometric functions becomes the domain of inverse trigonometric functions and restricted domain of trigonometric functions becomes range or principal value branch of inverse trigonometric functions.

Function	Domain	Principal value branch (Range)	Other possible range
$y=\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc.
$y=\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$	$[-\pi, 0], [\pi, 2\pi]$ etc.
$y=\tan^{-1}x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ etc.
$y=\sec^{-1}x$	$R-(-1, 1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$	$[-\pi, 0]-\left\{-\frac{\pi}{2}\right\}, [\pi, 2\pi]-\left\{\frac{3\pi}{2}\right\}$ etc.
$y=\text{cosec}^{-1}x$	$R-(-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]-\{0\}$	$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]-\{-\pi\}, \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]-\{\pi\}$
$y=\cot^{-1}x$	R	$(0, \pi)$	$(-\pi, 0), (\pi, 2\pi)$ etc.

Table for Domain, Range and Other Possible Range of Inverse Trigonometric Functions :

Graphs of Inverse Trigonometric Functions :

The graphs of inverse trigonometric functions with respect to line y = x are given in the following table

Function	**Graph** (By interchanging axes)	**Graph** (By mirror image)
$y=\sin^{-1}x$		

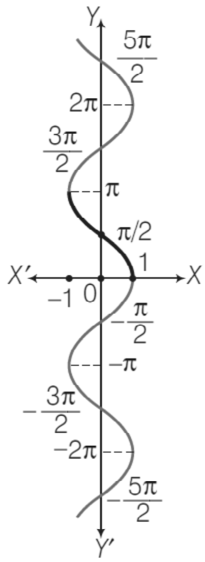
(1) $\sin^{-1}x$ is an odd function (Symmetric about origin)

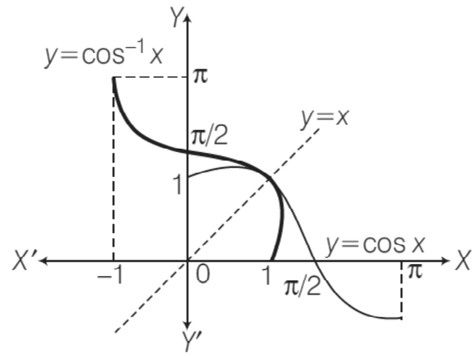
(2) $\sin^{-1}x$ is an increasing function in its domain.

(3) $\sin^{-1}x$ is a periodic function.

(4) $\sin^{-1}x$ is bounded in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$y=\cos^{-1}x$





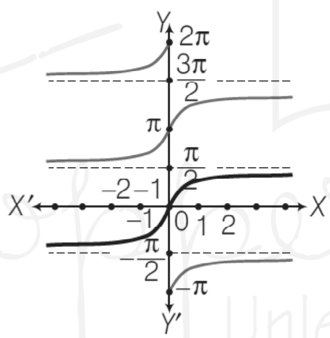
(1) $\cos^{-1}x$ is bounded in $[0,\pi]$.

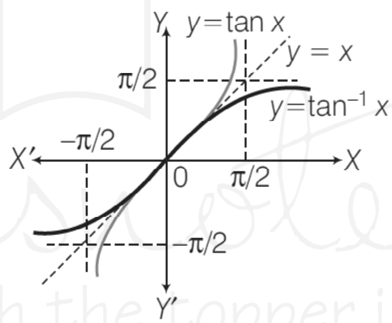
(2) $\cot^{-1}x$ is neither odd nor even function.

(3) $\cos^{-1}x$ is decreasing function in its domain.

(4) $\cos^{-1}x$ is a periodic function.

$y=\tan^{-1}x$



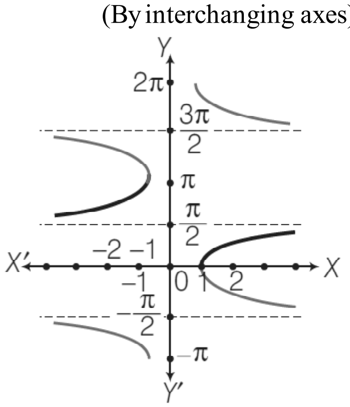
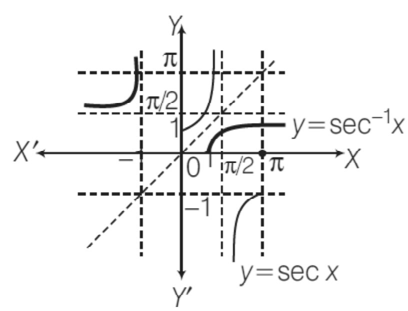


(1) $\tan^{-1}x$ is bounded in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(2) $\tan^{-1}x$ is an increasing function in its domain.

(3) $\tan^{-1}x$ is an increasing function in its domain.

(4) $\tan^{-1}x$ is a periodic function.

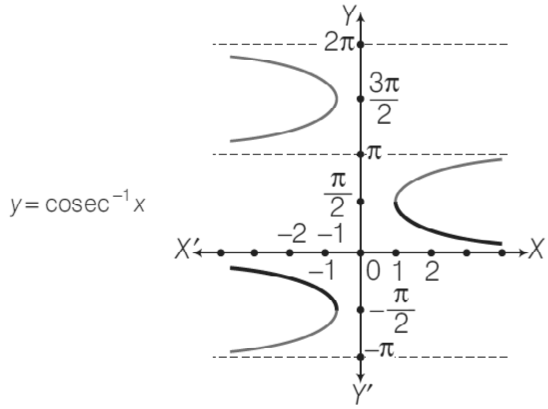
Function	**Graph** (By interchanging axes)	**Graph** (By mirror image)
$y=\sec^{-1}x$		

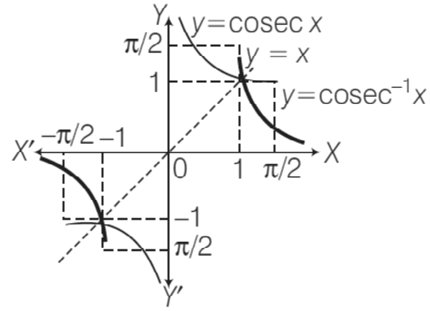
(1) $\sec^{-1}x$ is bounded in $[0,\pi]$

(2) $\sec^{-1}x$ is a neither odd nor even function.

(3) $\sec^{-1}x$ is an increasing function.

(4) $\sec^{-1}x$ is q periodic function.



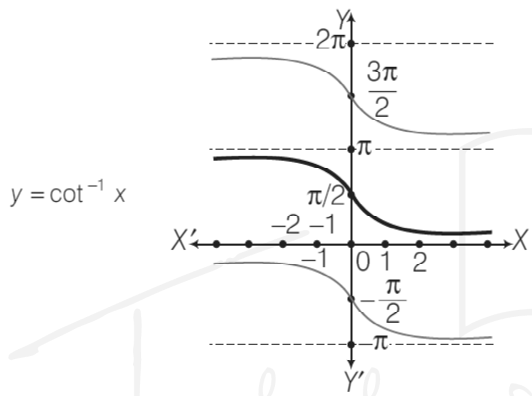


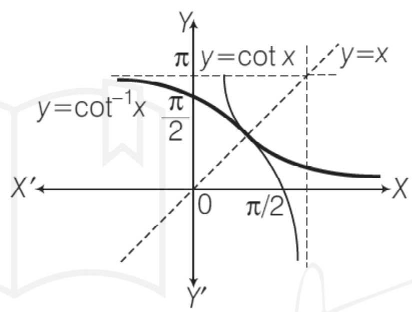
(1) $\text{cosec}^{-1} x$ bounded in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(2) cosec^{-1} is an odd function (Symmetric about origin)

(3) $\text{cosec}^{-1} x$ is decreasing function.

(4) cosec^{-1} is a periodic function.





(1) $\cot^{-1} x$ bounded in $(0,\pi)$

(2) $\cot^{-1}$ is neither odd nor even function.

(3) $\cot^{-1} x$ is decreasing function in its domain

(4) $\cot^{-1} x$ is a periodic function.

Elementary Properties of Inverse Trigonometric Functions :

Property I :

(i) $\sin^{-1}(\sin\theta) = \theta;\ \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta; \theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta; \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iv) $\text{cosec}^{-1}(\text{cosec}\ \theta) = \theta; \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \theta \neq 0$

(v) $\sec^{-1}(\sec\theta) = \theta; \theta \in [0, \pi], \theta \neq \frac{\pi}{2}$

(vi) $\cot^{-1}(\cot\theta) = \theta; \theta \in (0, \pi)$

Property II :

(i) $\sin^{-1}(-x) = -\sin^{-1}x;\ x \in [-1,1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x;\ x \in [-1,1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x;\ x \in R$

(iv) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x;\ x \in (-\infty,-1) \cup [1,\infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x;\ x \in (-\infty,-1) \cup [1,\infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x;\ x \in R$

Property III :

(i) $\sin^{-1}(-x) = -\sin^{-1}x;\ x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x;\ x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x;\ x \in R$

(iv) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x;\ x \in (-\infty,-1) \cup [1,\infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x;\ x \in (-\infty,-1) \cup [1,\infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x;\ x \in R$

Property IV :

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x;\ x \in (-\infty,-1) \cup [1,\infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x;\ x \in (-\infty,-1) \cup [1,\infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x; & \text{if } x > 0 \\ -\pi + \cot^{-1}x; & \text{if } x < 0 \end{cases}$

Property V :

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$; $x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$; $x \in R$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$; $x \in (-\infty, -1) \cup [1, \infty)$

Property VI :

(i)
$$\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; \\ \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \text{ or} \\ \text{if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; \\ \text{if } 0 < x, y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; \\ \text{if } 1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

(ii)
$$\sin^{-1} x - \sin^{-1} y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}; \\ \text{if } \ 1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ \text{or if } xy > 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}; \\ \text{if } 0 < x \le 1, -1 \le y \le 0 \text{ and } x^2 + y^2 > 1 \\ -\pi \ \ \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}; \\ \text{if } \ 1 < x < 0, 0 < y \le 1 \text{ and } x^2 + y^2 > 1 \end{cases}$$

Property VII :

(i) $\text{Cos}^{-1} x + \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}; & \text{if } -1 \le x, y \le 1 \text{ and } x + y \le 0 \\ 2\pi - \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}; & \text{if } -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$$

(ii) $\text{Cos}^{-1} x - \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}; & \text{if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}; & \text{if } -1 \le y \le 0, 0 < x \le 1 \text{ and } x \le y \end{cases}$$

Property IX :

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\frac{1}{x}\right), x \in (0, 1)$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} \quad \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\left(\frac{1}{x}\right)$

$= \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right), x \in (0, 1)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right)$

$= \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right), x \in (0, \infty)$

Property X :

(i) $$2\sin^{-1} x = \begin{cases} \sin^{-1}\left(2x\sqrt{1-x^2}\right); & \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right); & \text{if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right); & \text{if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

(ii) $$2\sin^{-1} x = \begin{cases} \cos^{-1}\left(2x^2 - 1\right); & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}\left(2x^2 - 1\right); & \text{if } -1 \le x \le 0 \end{cases}$$

(iii) $$2\tan^{-1} x = \begin{cases} \tan^{-1}\left(\frac{2x}{x - x^2}\right); & \text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right); & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right); & \text{if } x < -1 \end{cases}$$

Property XI :

(i) $$3\sin^{-1} x = \begin{cases} \sin^{-1}\left(3x - 4x^2\right); & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}\left(3x \quad 4x^3\right); & \text{if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}\left(3x \quad 4x^3\right); & \text{if } -1 \le x < -\frac{1}{2} \end{cases}$$

(ii) $$3\cos^{-1} x = \begin{cases} \cos^{-1}\left(4x^3 - 3x\right); & \text{if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}\left(4x^3 - 3x\right); & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}\left(4x^3 - 3x\right); & \text{if } -1 \le x < -\frac{1}{2} \end{cases}$$

(iii) $$3\tan^{-1} x = \begin{cases} \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right); & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right); & \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\frac{3x^3 - x^3}{1 - 3x^2}\right); & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$$

Property XII :

(i) $2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x > 1 \\ -\pi + \sin^{-1}\left(\dfrac{2x^3}{1+x^2}\right); & \text{if } x < -1 \end{cases}$

(ii) $2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } -\infty < x < 0 \end{cases}$

Some Important Results :

(i) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$
if $x > 0, y > 0, z > 0$ and $(xy + yz + zx) < 1$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$, then $xy + yz + zx = 1$

(iii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z = xyz$

(iv) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{\pi}{2}$, then $x^2 + y^2 + z^2 + 2xyz = 1$

(v) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, then
$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

(vi) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then $xy + yz + zx = 3$

(vii) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then $x^2 + y^2 + z^2 + 2xyz = 1$

(viii) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{3\pi}{2}$, then $xy + yz + zx = 3$

(ix) If $\sin^{-1}x + \sin^{-1}y = \theta$, then $\cos^{-1}x + \cos^{-1}y = \pi - \theta$

(x) If $\cos^{-1}x + \cos^{-1}y = \theta$, then $\sin^{-1}x + \sin^{-1}y = \pi - \theta$

(xi) If $\tan^{-1}x + \tan^{-1}y = \dfrac{\pi}{2}$, then $xy = 1$

(xii) If $\cot^{-1}x + \cot^{-1}y = \dfrac{\pi}{2}$, then $xy = 1$

(xiii) If $\cos^{-1}\dfrac{x}{a} + \cos^{-1}\dfrac{y}{b} = \theta$, then

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \sin^2\theta$$

Sample Questions CBSE

Basic Concepts

MCQ

1. In the given question, a statement of assertion (A) is followed by a statement of reason (R). Choose the correct answer out of the following choices.

Assertion (A) : The domain of the function $\sec^{-1} 2x$ is $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

Reason (R) : $\sec^{-1}(-2) = \frac{\pi}{4}$ **(2022-23)**

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true.

2. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ is equal to **(Term I, 2021-22)**

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) –1 (d) 1

3. $\sin(\tan^{-1}x)$, where $|x| < 1$, is equal to **(Term I, 2021-22)**

(a) $\frac{x}{\sqrt{1-x^2}}$

(b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{1+x^2}}$

(d) $\frac{x}{\sqrt{1+x^2}}$

4. Simplest form of **(Term I, 2021-22)**

$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}+\sqrt{1-\cos x}}\right), \pi < x < \frac{3\pi}{2}$ is

(a) $\frac{\pi}{4} - \frac{x}{2}$

(b) $\frac{3\pi}{2} - \frac{x}{2}$

(c) $-\frac{x}{2}$

(d) $\pi - \frac{x}{2}$

5. If $\tan^{-1} x = y$, then **(Term I, 2021-22)**

(a) $-1 < y < 1$

(b) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(c) $\frac{-\pi}{2} < y < \frac{\pi}{2}$

(d) $y \in \left\{\frac{-\pi}{2}, \frac{\pi}{2}\right\}$

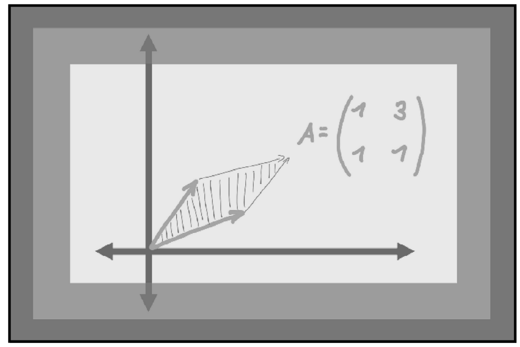
SAI

6. Find the value of $\sin^{-1}\left[\sin\left(\frac{13\pi}{7}\right)\right]$. **(2022-23)**

7. Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), \frac{-3x}{2} < x < \frac{\pi}{2}$ in the simplest form. **(2020-21)**

03

Matrices & Determinants



Matrix

Introduction :

A matrix is a rectangular arrangement of numbers (real or complex) which may be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots & a_{1n} \\ a_{11} & a_{22} & a_{23} & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3}\ldots & a_{mn} \end{bmatrix}$$

Matrix is enclosed by [ ] or ( ).

Compact form the above matrix is represented by $[a_{ij}]_{m\times n}$ or $A = [a_{ij}]$.

Element of a Matrix :

The numbers $a_{11}, a_{12}, \ldots$, etc., in the above matrix are known as the element of the matrix, generally represented as a_{ij} , which denotes element in ith row and jth column.

Order of a Matrix :

In above matrix has m rows and n columns, then A is of order m × n.

Types of Matrices :

(i) Row Matrix : A matrix having only one row and any number of columns is called a row matrix.

(ii) Column Matrix : A matrix having only one column and any number of rows is called column matrix.

(iii) Null/Zero Matrix : A matrix of any order, having all its elements are zero, is called a null/zero matrix, i.e. $a_{ij} = 0, \forall\ i, j$.

(iv) Square Matrix : A matrix of order m × n, such that m = n, is called square matrix.

(v) Diagonal Matrix : A square matrix $A = [a_{ij}]_{m\times n}$ is called a diagonal matrix, if all the elements except those in the leading diagonals are zero, i.e. $a_{ij} = 0$ for $i \neq j$. It can be represented as
$A = \text{diag}\ [a_{11}\ a_{22} \ldots a_{nm}]$.

(vi) Scalar Matrix :
A square matrix in which every non-diagonal element is zero and all diagonal elements are equal, is called scalar matrix, i.e. in scalar matrix, $a_{ij} = 0$, for $i \neq j$ and $a_{ij} = k$, for $i = j$.

(vii) Unit/Identity Matrix : A square matrix, in which every non-diagonal element is zero and every diagonal element is 1, is called unit matrix or an identity matrix,

$$\text{i.e. } a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ 1, & \text{when } i = j \end{cases}$$

(viii) Rectangular Matrix : A matrix of order m × n,such that $m \neq n$, is called rectangular matrix.

(ix) Horizontal Matrix : A matrix in which the number of rows is less than the number of columns, is called horizontal matrix.

(x) Vertical Matrix : A matrix in which the number of rows is greater than the number of columns, is called vertical matrix.

(xi) Upper Triangular Matrix : A square matrix $A = [a_{ij}]_{n\times n}$ is called a upper triangular matrix, if $a_{ij} = 0, \forall\ i > j$.

(xii) Lower Triangular Matrix : A square matrix $A = [a_{ij}]_{n\times n}$ is called a lower triangular matrix, if $a_{ij} = 0, \forall\ i > j$.

(xiii) Submatrix A matrix : which is obtained from a given matrix by deleting any number of rows or columns or both is called a submatrix of the given matrix.

(xiv) Equal Matrices Two matrices : A and B are said to be equal, if both having same order and cor responding elements of the matrices are equal.

(xv) Principal Diagonal of a Matrix : In a square matrix, the diagonal from the first element of the first row to the last element of the last row is called the principal diagonal of a matrix.

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 5 \\ 1 & 1 & 2 \end{bmatrix}$, then pricipal diagonal of A is 1, 6, 2.

(xvi) Singular Matrix : A square matrix A is said to be singular matrix, if determinant of A denoted by det (A) or |A| is zero, i.e. |A| = 0, otherwise it is a non-singular matrix.

Algebra of Matrices :

1. *Addition of Matrices :*

Let A and B be two matrices each of order m × n. Then, the sum of matrices A + B is defined only if matrices A and B are of same order.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m\times n}$.

Then $A + B = [aij + b_{ij}]_{m \times n}$.

Properties of Addition of Matrices :

If A, B and C are three matrices of order m × n, then

(i) Commutative Law : A + B = B + A

(ii) Associative Law : (A + B) + C = A + (B + C)

(iii) Existence of Additive Identity A zero matrix (0) of order m × n (same as of A), is additive identity, if A + 0 = A = 0 + A

(iv) Existence of Additive Inverse : If A is a square matrix, then the matrix (– A) is called additive inverse, if A + (– A) = 0 = (–A) + A

(v) Cancellation Law : $A + B = A + C \Rightarrow B = C$ [left cancellation law]

$B + A = C + A \Rightarrow B = C$ [right cancellation law]

2. *Subtraction of Matrices :*

Let A and B be two matrices of the same order, then subtraction of matrices, A – B, is defined as

$A - B = [a_{ij} - b_{ij}]_{m\times n}$,

where $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

3. *Multiplication of a Matrix by a Scalar :*

Let $A = [a_{ij}]_{m \times n}$ be a matrix and k be any scalar. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA, given as

$kA = [ka_{ij}]_{m \times n}$

Properties of Scalar Multiplication :

If A and B are two matrices of order m × n, then

(i) $k(A + B) = kA + kB$

(ii) $(k_1 + k_2)A = k_1A + k_2A$

(iii) $k_1k_2A = k_1(k_2A) = k_2(k_1A)$

(iv) $(–k)A = –(kA) = k(–A)$

4. *Multiplication of Matrices :*

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices such that the number of columns of A is equal to the number of rows of B, then multiplication of A and B is denoted

by AB, is given by $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$,

where c_{ij} is the element of matrix C and C = AB.

e.g. if $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, then

$$AB = \begin{bmatrix} a_1b_2 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{bmatrix}.$$

Properties of Multiplication of Matrices :

(i) Associative Law : (AB)C = A(BC)

(ii) Existence of Multiplicative Identity : A· I = A = I· A, where, I is called multiplicative Identity.

(iii) Distributive Law : A(B + C) = AB + AC

(iv) Cancellation Law : If A is non-singular matrix, then

$AB = AC \Rightarrow B = C$ [left cancellation law]

$BA = CA \Rightarrow B = C$ [right cancellation law]

(v) Zero Matrix as the Product of Two Non-zero :

Matrices AB = O, does not necessarily simply that A = O or B = O or both A and B = O.

Note :

Multiplication of diagonal matrices of same order will be commutative.

Important Points to be Remembered :

(i) If A and B are square matrices of the same or der, say n, then both the product AB and BA are defined and each is a square matrix of order n.

(ii) In the matrix product AB, the matrix A is called premultiplier (prefactor) and B is called postmultiplier (postfactor).

(iii) The rule of multiplication of matrices is row columnwise (or $\rightarrow \downarrow$ wise) the first row of AB is obtained by multiplying the first row of A with first, second, third,... columns of B respectively; similarly second row of A with first, second, third, ... columns of B, respectively and so on.

Positive Integral Powers of a Square Matrix :

Let A be a square matrix. Then, we can define

(i) $A^{n+1} = A^n \cdot A$, where $n \in N$.

(ii) $A^m \cdot A^n = A^{m+n}$.

(iii) $(A^m)^n = A^{mn}$, $\forall$ m, n $\in$ N

Matrix Polynomial :

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$. Then,

$f(A) = a_0A^n + a_1A^{n-2} + ...a_n I_n$ is called the matrix polynomial.

Transpose of a Matrix :

Let $A = [a_{ij}]_{m \times n}$, be a matrix of order $m \times n$. Then, the $n \times m$ matrix obtained by interchanging the rows and columns of A is called the transpose of A and is denoted by A' or A^T.

$A' = A^T = [aij]_{n \times m}$

Properties of Transpose :

For any two matrices A and B of suitable orders,

(i) $(A')' = A$ (ii) $(A \pm B)' = A' \pm B'$

(iii) $(kA)' = kA'$ (iv) $(AB)' = B'A'$

(v) $(A^n)' = (A')^n$ (vi) $(ABC)' = C'B'A'$

Symmetric and Skew-Symmetric Matrices :

(i) A square matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric, if $A' = A$. i.e. $a_{ij} = a_{ji}$, $\forall$ i and j.

(ii) A square matrix A is said to be skew-symmetric, if $A' = -A$, i.e. $a_{ij} = -a_{ji}$, $\forall$ i and j.

Properties of Symmetric and Skew-symmetric Matrices :

(i) Elements of principal diagonals of a skew-sym metric matrix are all zero. i.e. $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$ or $a_{ii} = 0$, for all values of i.

(ii) If A is a square matrix, then
(a) $A + A'$ is symmetric.
(b) $A - A'$ is skew-symmetric matrix.

(iii) If A and B are two symmetric (or skew-symmetric) matrices of same order, then $A + B$ is also symmetric (or skew-symmetric).

(iv) If A is symmetric (or skew-symmetric), then kA(k is a scalar) is also symmetric (or skew-symmetric) matrix.

(v) If A and B are symmetric matrices of the same order, then the product AB is symmetric, if $BA = AB$.

(vi) Every square matrix can be expressed uniquely as the sum of a symmetric and a skew-symmet ric matrix.
i.e. Matrix A can be written as

$$\frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

(vii) The matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric matrix.

(viii) All positive integral powers of a symmetricmatrix are symmetric.

(ix) All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric are symmetric matrix.

(x) If A and B are symmetric matrices of the same order, then

(a) $AB - BA$ is a skew-symmetric and

(b) $AB + BA$ is symmetric.

(xi) For a square matrix A, AA' and $A'A$ are symmetric matrix.

Elementary Operations

(Transformations of a Matrix)

Any one of the following operations on a matrix is called an elementary transformation.

(i) Interchanging any two rows (or columns), de noted by $R_i \to R_j$ or $C_i \leftrightarrow C_j$.

(ii) Multiplication of the element of any row (or column) by a non-zero scalar quantity and de noted by $R_i \to kR_i$ or $C_i \to kC_j$.

(iii) Addition of constant multiple of the elements of any row to the corresponding element of any other row, denoted by $R_i \to R_i + kR_i$ or $C_i \to C_j + kC_j$.

Elementary Matrix :

A matrix obtained from an identity matrix by a single elementary operation is called an elementary matrix.

Equivalent Matrix :

Two matrices A and B are said to be equivalent, if one can be obtained from the other by a sequence of elementary transformation.

The symbol $\approx$ is used for equivalence.

Trace of a Matrix :

The sum of the diagonal elements of a square matrix A is called the trace of A, denoted by trace (A) or tr (A).

Conjugate of a Matrix

The matrix obtained from a matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex number is called conjugate of A and is denoted by A.

Properties of Conjugate of a Matrix :

Let A and B are two matrices of order $m \times n$ and k be a scalar, then

(i) $\overline{(\overline{A})} = A$ (ii) $\overline{(A + B)} = \overline{A} + \overline{B}$

(iii) $\overline{(AB)} = \overline{A}\,\overline{B}$ (iv) $\overline{(kA)} = k\overline{A}$

(v) $\overline{(A^n)} = (\overline{A})^n$

Transpose Conjugate of a Matrix :

The transpose of the conjugate of a matrix A is called transpose conjugate of A and is denoted by A^θ or A^*,

i.e. $\overline{(A')} = (\overline{A})' = A^\theta$ or A^*